LECTURE TWO: REVIEW OF SOME BASICS OF DYNAMICAL SYSTEMS

Consider plants in a small garden. Let $x = 0$ represent an empty garden and $x = 1$ represent a "full" garden. Suppose that the birth rate is equivalent to the population size, and the death rate is equivalent to the square of the population size. We can model this as follows,

$$\dot{x} = f(x) = x - x^2. \tag{1}$$

First we wish to find the fixed points, $x_* = 0, 1$. Now we can draw the vector field, as done in class. Notice that we allow $x > 1$, but this means that the garden is over capacity and plants die off due to crowding. Recall that we may determine the stability without graphical means by taking the derivative,

$$f'(x_*) = 1 - 2x_* \Rightarrow f'(0) > 0,\ f'(1) < 0.$$

So, $x_* = 0$ is unstable and $x_* = 1$ is stable. However, this fails for fixed points where $f'(x_*) = 0$. This is called a nonhyperbolic fixed point. A hyperbolic fixed point is the case in our example where $f'(x_*) \neq 0$.

Consider further an animal that eats the plants at a $1 : 1$ rate is introduced into the system. Suppose that the birth rate is equivalent to the population size of the plants, and the death rate is equivalent to its population size. Further, assume the animals' poop acts like a fertilizer, effectively doubling the plants' population growth rate. We can model this as follows,

$$\begin{aligned} \dot{x} &= 2x - x^2 - y, \\ \dot{y} &= x - y. \end{aligned} \tag{2}$$

First we look for the nullclines, i.e. curves such that $\dot{x} = 0$ or $\dot{y} = 0$,

$$\begin{aligned} \dot{x} = 0 &\Rightarrow y = 2x - x^2, \\ \dot{y} = 0 &\Rightarrow y = x. \end{aligned}$$

Recall that fixed points are the intersection points of these nullclines, so our fixed points are, $(x_*, y_*) = (0, 0)$, $(1, 1)$. Now we find the Jacobian in order to find the eigenvalues and eigenvectors for the respective fixed points. The general Jacobian for this system is,

$$J(x_*, y_*) = \begin{pmatrix} 2 - 2x_* & -1 \\ 1 & -1 \end{pmatrix} \tag{3}$$

Then respectively we have,

$$J(0, 0) = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix};\ J(1, 1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{4}$$

Then the eigenvalues for $(x_*, y_*) = (0, 0)$ are,

$$J(x_*, y_*) = \begin{vmatrix} 2 - 2\lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

And the eigenvalues for $(x_*, y_*) = (1, 1)$ are,

$$J(x_*, y_*) = \begin{vmatrix} -\lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

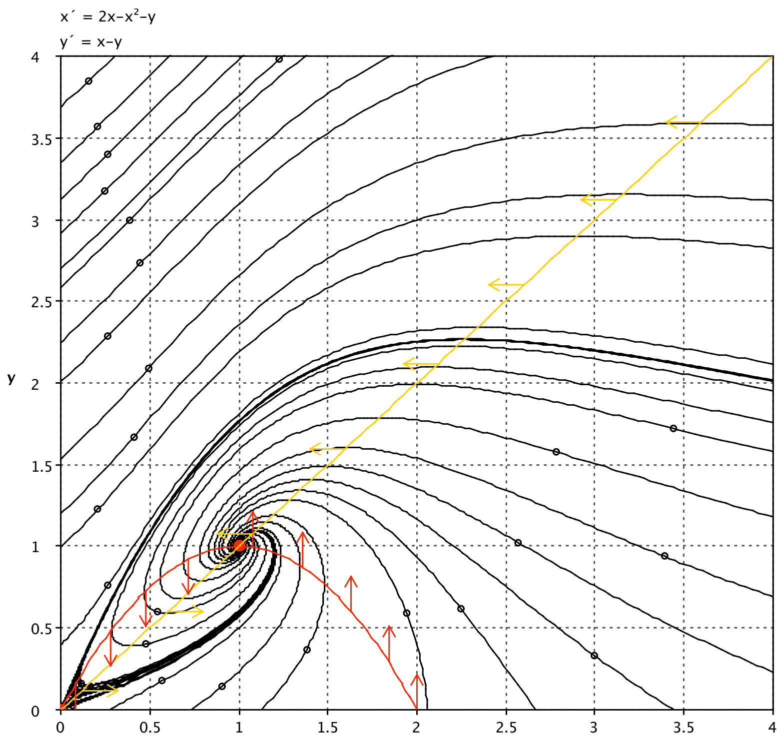
Then the respective eigenvectors are,

$$\begin{pmatrix} \frac{3}{2} \mp \frac{\sqrt{5}}{2} & -1 \\ 1 & -\frac{3}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ \frac{3}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix}$$

And

$$\begin{pmatrix} \frac{1}{2} \mp i\frac{\sqrt{3}}{2} & -1 \\ 1 & -\frac{2}{2} \mp i\frac{\sqrt{3}}{2} \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ \frac{1}{2} \mp i\frac{\sqrt{3}}{2} \end{pmatrix}$$

From all this information we can manually sketch the phase plane. However, for the sake of illustration I have done this on pplane,



Moving on, lets recall what potentials are. Consider the system $\dot{x} = f(x)$, then the potential for this system is $V(x)$ such that $f(x) = -dV/dx$. Also recall, $dV/dt = (dV/dx)(dx/dt) = -(dV/dx)^2$ because $\dot{x} = -dV/dx$. So, $V(t)$ decreases along trajectories, i.e. things go from higher to lower potential.

Now we can discuss conservative systems. A system where there is a quantity that is invariant along the flow. A special type of conservative system is called a Hamiltonian system, which is just a reformulation of $F = ma$ for point particles in a force field. Recall, if we have a system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ and there exists an H such that $\partial H/\partial y = f(x, y)$ and $\partial H/\partial x = -g(x, y)$ then the system is a Hamiltonian system and the conserved quantity H is a Hamiltonian. Then the level sets $H = E(x, y)$ are invariant. Example: Pendulum. In class I quickly sketched the phase plane of this.

Finally, recall that sometimes we are interested in flows on a circle. These systems are of the kind $\dot{\theta} = f(\theta)$. Notice that we can transform this system into $\dot{x} = f(x)$ such that $f(0) \equiv f(1)$, i.e. we need only identify $x = 0$ with $x = 1$. This will be useful to us when we study Peixoto's structural stability theorem.