LECTURE THREE: WHAT IS CHAOS?

Typing “chaos” into google yields 240,000,000 results in .28 seconds.

Google defines chaos as complete disorder and confusion; behavior so unpredictable as to appear random, owing to great sensitivity to small changes in conditions.

Wikipedia says chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions.

Lets see what some experts have to say about chaos.

Lorenz said, ”Chaos: when the present determines the future, but the approximate present doesn’t approximately determine the future” We should keep in mind that he is thinking of a climate system.

Poincare said, ”It may happen that slight variations in the initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon.”

These are philosophical definitions, but how about a mathematical one? Meiss defines it in his book on differential dynamical systems,

Definition 1. A flow φ is chaotic on a compact invariant set $\mathbb{X}$ if φ is transitive and exhibits sensitive dependence on $\mathbb{X}$.

How about from a classical mechanics point of view? Taylor writes, ”This erratic, nonperiodic long-term behavior is one of the defining characteristics of chaos. The other defining characteristic is the phenomenon called sensitivity to initial conditions.”

Now, lets get back to some mathematical definitions. Glendining loosely defines a chaotic solution as aperiodic but bounded and nearby trajectories seperate rapidly. And formally defines it as,

Definition 2. A continuous map $f : \mathbb{I} \to \mathbb{I}$ is chaotic if and only if f^n has a horseshoe for some $n \geq 1$.

This definition would be much to technical for us at the moment so we will skip over it for now.

Robinson gives a similar definition as Meiss except for maps, but he says Devaney (1989) gave an explicit definition of a chaotic invariant set in an attempt to clarify the notion of chaos. To our two assumptions he adds the assumption that the periodic points are dense in Y (an invariant set). Although this last property is satisfied by “uniformly hyperbolic” maps like the quadratic map, it does not seem that this condition is at the heart of the idea that the system is chaotic.

So we see here, that even the experts disagree on the definition of chaos. Strogatz puts it very nicely, “No definition of the term chaos is universally accepted yet, but almost everyone would agree on the three ingredients used in the following working definitions:”

Definition 3. Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Lets go back to Meiss's formal definition and dissect it. The only two terms that we may not be familiar with are "sensitive dependence" and "transitive".

Definition 4. A flow φ exhibits sensitive dependence on an invariant set $\mathbb{X}$ if there is a fixed r such that for each $x \in \mathbb{X}$ and any $\varepsilon > 0$, there is a nearby $y \in B_\varepsilon(x) \cup \mathbb{X}$ such that $|\varphi_t(x) - \varphi_t(y)| > r$ for some $t \geq 0$.

This doesn't mean that all pairs of nearby points act like this, but we can find points that do.

Definition 5. A flow φ is topologically transitive on an invariant set $\mathbb{X}$ if for every pair of nonempty, open sets $\mathbb{U}, \mathbb{V} \subset \mathbb{X}$ there is a $t > 0$ such that $\varphi_t(\mathbb{U}) \cup \mathbb{V} \neq \emptyset$

Basically the flow will wander all over $\mathbb{X}$.

Furthermore, if we transform a chaotic system from one set to another we would like it to still be chaotic. The following theorem outlines the conditions under which this is possible.

Theorem 1. *Suppose $\varphi_t : \mathbb{X} \to \mathbb{X}$ and $\psi_t : \mathbb{Y} \to \mathbb{Y}$ are flows, $\mathbb{X}$ and $\mathbb{Y}$ are compact, and φ is chaotic on $\mathbb{X}$. The if ψ is conjugate to φ, it too is chaotic.*

Now, what type of systems can not be chaotic? First lets define an omega limit set.

Definition 6. The ω-limit set of a point x is,

$$\omega(x) := \{y | \varphi_{t_k}(x) \to y \text{ as } t_k \to \infty\}.$$

We have a similar definition for α-limit sets.

Now, in 1-D flows we can't have chaos because $\omega(x)$ can only be a fixed point.

In 2-D flows we can't have chaos because of the Poincaré-Bendixson theorem,

Theorem 2. *Let D be a simply connected subset of $\mathbb{R}^2$ and φ be a flow in D. Suppose that the forward orbits of some $p \in D$ is contained in a compact set and that $\omega(p)$ contains no fixed points. The $\omega(p)$ is a periodic orbit.*