LECTURE FOUR: MAPS! WHAT ARE THEY GOOD FOR!?

Maps are more natural models to certain physical systems, namely systems with natural recurrence. they can be easier to analyze sometimes. They are much easier to simulate (analyze numerically). This is because ODEs require complicated - slow numerical schemes, and sometimes that doesn't even work. Maps only require loops carrying out a couple of arithmetic operations.

Definition 1. Maps are time discrete dynamical systems represented by the recurrence relation

$$x_{m+1} = f(x_m);\ x_m \in \mathbb{R}^n \tag{1}$$

One great thing about maps is that we can create very simple examples of chaos since we aren't restricted by the dimensions of the dynamical system. Lets recall what some properties of maps are. We define the fixed point as $x_* = f(x_*)$. This is done because $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$. Consider $x_{n+1} = x_n^2$, to find the fixed point we do, $x_*^2 - x_* = 0$, so our fixed points are $x_* = 0, 1$.

For stability all eigenvectors corresponding to $|\lambda| < 1$ are the stable direction, all eigenvectors corresponding to $|\lambda| > 1$ are the unstable direction, and all eigenvectors corresponding to $|\lambda| = 1$ form the center subspace. It's easy to prove this in 1-D.

Proof. Consider $x_n = x_* + \xi_n$, then $x_* + \xi_{n+1} = x_{n+1} = f(x_* + \xi_n)$. We can find the Taylor series of f, $f(x + \xi_n) = f(x_*) + f'(x_*)\xi_n + o(\xi_n^2)$. Since $f(x_*) = x_*$, $\xi_{n+1} = f'(x_*)\xi_n + o(\xi_n^2)$. Suppose $\xi_{n+1} \approx f'(x_*)\xi_n$, and let $\lambda = f'(x_*)$. Now, since $\xi_1 = \lambda\xi_0$, by induction $\xi_n = \lambda^n\xi_0$, where ξ_0 is a constant. Therefore, if $|\lambda| < 1$, $\xi_n \to 0$ exponentially fast as $n \to \infty$; if $|\lambda| > 1$, $\xi_n \to 0$ exponentially fast as $n \to -\infty$; if $|\lambda| = 1$, ξ_n grows subexponentially, i.e. $o(\xi_n^2)$ matter. □

Now, consider the logistic map: $x_{n+1} = rx_n(1 - x_n)$. This comes from similar ideas to the logistic ODE, but $x_n = 1$ represents absolute capacity, i.e. if we have plants, there are so many that the soil becomes so devoid of nutrition that it can never sustain life again. The fixed point for this map is $x_* = (r-1)/r$. For stability we take the derivative $f'(x_*) = r - 2rx_* = 2 - r$, so x_* is unstable for $r < 1$, stable for $1 < r < 3$, and unstable for $r > 3$. For the stable case there's no ambiguity but for the unstable case we don't know in which way it's unstable, and for the borderline cases we don't even know the stability.

In order to rectify this, we may use cobweb plots to help us illustrate the global behavior of a system at a glance. We did examples of cobwebs in class.

As with continuous systems, discrete systems experience bifurcations and contain periodic orbits. Lets not discuss bifurcations too much, but you should read up on them on pg. 358. What is a periodic orbit for a discrete system?

Definition 2. We say a point $\hat{x}$ is contained in a k-cycle if $\hat{x} = f^k(\hat{x})$ and $\hat{x} \neq f^{k-1}(\hat{x})$, where f^k is the k^{th} iteration of $x_{n+1} = f(x_n)$.

Consider the map $x_{n+1} = 2 - x_n$, then $f^2(\hat{x}) = f(f(\hat{x})) = 2 - (2 - \hat{x}) = \hat{x}$. So every point for this map is in some 2-cycle. This is a trivial example, so lets look at the logistic map. $f^2(x) - x = r^2 x(1-x)[1 - rx(1-x)] - x = 0$. Factoring out x and $x - (r-1)/r$ gives, $\hat{x} = (r + 1 \pm \sqrt{(r-3)(r+1)})/2r$. So these are the two points in the two cycle. Notice that for certain values of r the discriminant is negative, and hence $\hat{x}$ is not real, therefore for those values of r there is no two cycle.

For the logistic map, from $r = 3$ the fixed point starts to bifurcate into a 2-cycle, then a 4-cycle, then eventually to chaos. This is called period doubling. There's more on it on pg. 353.

A useful tool to quantify "sensitive dependence" is Liapunov exponents. If δ_0 is the initial separation, $|\delta_n| = |\delta_0| e^{n\lambda}$, then

$$\lambda = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{2}$$

The derivation for this is on pg. 366. We derive the formula for the Liapunov exponents of the logistic map in class.