LECTURE FIVE: POINCARÉ MAPS AND STRUCTURAL STABILITY

Lecture Five Part I: Poincaré Maps. First lets write down a few key definitions.

Definition 1. Suppose $\varphi_t(x_*)$ is a periodic orbit of $\dot{x} = f(x)$, where f is continuously differentiable, of period T,

$$\Gamma = \{x \in \mathbb{R}^n : x = \varphi_t(x_*), 0 \leq t \leq T\}. \tag{1}$$

Definition 2. Let Σ be the hyperplane orthogonal to Γ at x_*,

$$\Sigma = \{x \in \mathbb{R}^n : (x - x_*) \cdot f(x_*) = 0\}. \tag{2}$$

Theorem 1. *There is an $\varepsilon > 0$ and a unique function $\tau(x)$, which is continuously differentiable for $x \in B_\varepsilon(x_*)$ such that $\tau(x_*) = T$ and $\varphi_{\tau(x)}(x) \in \Sigma$ for all $x \in B_\varepsilon(x_*)$.*

Definition 3. For $x \in B_\varepsilon(x_*) \cup \Sigma$, the function $P(x) = \varphi_{\tau(x)}(x)$ is called a Poincaré map for Γ at x_*.

Ex Consider the following ODE,

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)\\ \dot{y} &= x + y(1 - x^2 - y^2)\end{aligned} \tag{3}$$

We notice that we can write this in polar coordinates via the transformation, $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}\dot{r} &= r(1 - r^2)\\ \dot{\theta} &= 1\end{aligned} \tag{4}$$

Now, we notice that the fixed points of the r equation correspond to limit cycles of the x, y equation. Therefore, we have a limit cycle when $r = 1$, which corresponds to $(x, y) = (\cos t, \sin t)^T$. We can solve this via separation,

$$\begin{aligned}r &= \sqrt{1 + \left(\frac{1}{r(0)^2} - 1\right) e^{-2t}},\\ \theta &= t + \theta(0).\end{aligned}$$

Let $\theta(0) = \theta_0$, $r(0) = r_0$, and Σ be the ray $\theta = \theta_0$ through the origin. Then, Σ is orthogonal to Γ, which has a period of $T = 2\pi$, so the Poincaré map is

$$r_{n+1} = P(r_n) = \sqrt{1 + \left(\frac{1}{r_n^2} - 1\right) e^{-4\pi}}. \tag{5}$$

Notice that $r_* = 1$. We wish to use the Poincaré map to find the stability of this limit cycle, so we take the derivative,

$$P'(r_n) = e^{-4\pi} r_n^{-3} \left[1 + \left(\frac{1}{r_n^2} - 1 \right) e^{-4\pi} \right]^{-3/2} \Rightarrow |P'(1)| = e^{-4\pi} < 1.$$

In general it's impossible to write down the Poincaré map explicitly, but there are theorems to help our analysis.

Lecture Five Part II: Structural Stability. In class I basically outlined the ideas from my paper. You can access it here: `http://arxiv.org/abs/1306.0436`

Instead of rewriting it here, I will only go through the details of some of the examples I provided in the paper.

Ex **(Homeomorphism)** Consider the function $h : (0, \infty) \to (0, 1)$ defined by $h = \left(1 + x^2\right)^{-1}$. Lets prove that this is a homeomorphism.

Proof. First lets show the inverse exists, i.e. the inverse is well defined on the codomain. We compute the inverse to be $h^{-1} = \sqrt{1 - 1/x}$, which is defined for all $x \in (0, 1)$.

Next lets show that it is injective (i.e. one-to-one). Suppose $h(a) = h(b)$, then $(1 + a^2)^{-1} = (1 + b^2)^{-1} \Rightarrow 1 + b^2 = 1 + a^2 \Rightarrow b^2 = a^2$, and since $a, b \in (0, \infty)$, $b = a$.

Now lets show that it is surjective (i.e. onto). Consider $y \in (0, 1)$, then if $h(x) = y$, $x = \sqrt{1 - 1/y} \in (0, \infty)$.

Finally, we show that it is continuous. We see that $h(x = c) = \left(1 + c^2\right)^{-1} \in (0, 1)$, and $\lim_{x \to c} h(x) = h(c) \in (0, 1)$.

Similarly the inverse is injective, surjective, and continuous. □

Ex **(Topological Equivalence)** Consider the dynamical systems $\dot{\theta} = \sin \theta$ and $\dot{\varphi} = \cos \varphi$. Lets prove that these are topologically equivalent on $\mathbb{S}^1$.

Proof. Notice that if $\varphi = \theta - \pi/2$, our functions are equivalent. So, our homeomorphism is, $h : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $h(\theta) = \theta - \pi/2$. □