LECTURE SIX: CHAOTIC SCATTERING

In the mid 1800s John Scott Russel saw a boat stop in water, which caused a raised mass of water translate through the canal. Legend has it that he chased it for two miles and the wave kept going. This is a very specific case of a solitary wave, but mathematically they are solutions to non-linear evolutionary PDEs that translate at a constant speed while maintaining its profile.

We drew a graph of the solution to the Korteweg-de Vries in class. We also spoke briefly about the nonlinear Schrodinger equation and the Sine-Gordon equation.

In more recent years mathematicians have been studying the φ^4 equation: $\varphi_{tt} - \varphi_{xx} + \varphi - \varphi^3 = 0$. When studying the relation of the input velocity to the output velocity of the waves they noticed that the output velocity is very sensitively dependent on the input velocity, and thus was dubbed chaotic scattering.

This PDE can be reduced to a system of two ODEs that approximate the behavior and reproduce the v_{in}-v_{out} plots qualitatively.

$$\begin{aligned} m\ddot{X}(t) + U'(X) + cAF'(X) = 0, c = \varepsilon \ll 1 \\ \ddot{A}(t) + \omega^2 A + cF(X) = 0; \\ U(X) = e^{-2X} - e^{-X};\ F(X) = e^{-X}. \end{aligned} \tag{1}$$

We wish to construct a mechanical analog of this. Consider a ball rolling on a surface $z = h(x, y)$ where x corresponds to X and y corresponds to A. We can derive the equations of motion by using the Lagrangian: $\mathcal{L} = T - U$. Here the potential energy is simply $U = mgh(x, y)$ and the kinetic energy is the translational plus rotational kinetic energy of a sphere,

$$T = \frac{7}{5}\frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + (h_x\dot{x} + h_y\dot{y})^2\right].$$

Then we employ the Euler-Lagrange equations,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \\ \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \end{aligned}$$

If h varies slowly, i.e. all nonlinear h derivatives are approximately zero, we get

$$\begin{aligned} \ddot{x} + \frac{5}{7}gh_x(x, y) = 0, \\ \ddot{y} + \frac{5}{7}gh_y(x, y) = 0. \end{aligned} \tag{2}$$

Now we need to figure out what h needs to be to match our original ODE, and we get

$$h(x,y) = \frac{7}{5g}\left[U(x) + \frac{\omega^2}{2}y^2 + \varepsilon F(x)y\right] = \eta(e^{ax} - e^{bx}) + cy^2 + \varepsilon y e^{dx}; \tag{3}$$

$a = 0.389\text{cm}^{-1}$, $b = 0.306\text{cm}^{-1}$, $c = 0.0764\text{cm}^{-1}$, $d = 0.306\text{cm}^{-1}$, $\eta = 3.27\text{cm}$, $\varepsilon = 0.25$.

In addition we have the initial conditions (because of our ramp):

$$\dot{y}(0) = y(0) = x(0) = 0,\ \dot{x}(0) = v_{\text{in}} = \sqrt{10gh/7}. \tag{4}$$

Here use as accurate a value for g as possible. Also, x, y, and h are in cm, and g is in cm/s^2. Don't mess up the units or you'll get wacky results.

Now lets look at a the bit-shift map,

$$x_{n+1} = 2x_n (\mod 1). \tag{5}$$

Note that $x_n \in [0,1]$ and either $x_n \in I_0$ or $x_n \in I_1$. It's quite natural to let x_n be binary, i.e.

$$x_n = 0.\alpha_0\alpha_1\cdots\alpha_k\cdots;\ \alpha_k = 0,1. \tag{6}$$

Then,

$$x_{n+1} = (\alpha_0.\alpha_1\cdots\alpha_k\cdots)(\mod 1) = 0.\alpha_1\alpha_2\cdots\alpha_k\cdots.$$