LINEAR ALGEBRA

- **Know how to multiply matrices!**
- Know how to take the determinant of a 2×2 matrix.
- Solving 2×2 eigenvalue problems $Ax = \lambda x$: Find the eigenvalues, then find the corresponding eigenvectors.
- Know the three types of eigenvalues for 2×2 matrices:
  - Distinct Real Eigenvalues (i.e. positive discriminant)
  - Repeated Real Eigenvalues (i.e. zero discriminant)
  - Complex Conjugate Eigenvalues (i.e. negative discriminant)

MATRIX ODES

- Solve for the eigenvalues and eigenvectors of a 2×2 matrix ODE.
- Know the general form of solutions for the three cases: Distinct Real, Repeated Real, and Complex Conjugate eigenvalues.
- Solve an IVP.

DYNAMICAL SYSTEMS

- Find fixed points.
- Linearize about fixed points (i.e. the first nonconstant term in the Taylor series of the nonlinear function).
- Find the eigenvalues for each linearized matrix. No need to find eigenvectors in this class, although they are important.
- State the stability of each fixed point with proper reasoning.
- Sketch a complete phase portrait: fixed points and important trajectories. No need to find nullclines in this class, although they too are important.

1. Heat Equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \tag{1}$$

- Plug in separation of variables Ansatz into the equation to get the ODEs

$$T' = -K\lambda^2 T; \qquad X'' + \lambda^2 X = 0. \tag{2}$$

- The T solution is easiest: $T = e^{-K\lambda^2 t}$.
- The X equation gives us a Sturm–Liouville problem:
  - Solve for $\lambda = 0$: $X = c_1 x + c_2$. Plug in BCs to solve for constants.
  - Solve for $\lambda \neq 0$: $X = A\cos\lambda x + B\sin\lambda x$. Plug in BCs to solve for one constant and the eigenvalue λ^2.
- Write the general solution $u(x,t) = TX$ as a Fourier series.
- Plug in the **one** initial condition to find the other constants for the complete solution.

1.1. Nonhomogeneous BCs.

- Find the equilibrium solution $u_*(x)$ by solving $u_{xx} = 0$.
- Plug in the nonhomogeneous BCs to u_* to solve for the constants.
- Make the change of variables $v(x,t) = u(x,t) - u_*(x)$.
- Plug back into the original nonhomogeneous PDE to get

$$\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial x^2}; \qquad v(0,t) = v(L,t) = 0; \qquad v(x,0) = u(x,0) - u_*(x). \tag{3}$$

2. Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4}$$

- Plug in separation of variables Ansatz into the equation to get the ODEs

$$T'' + c^2\lambda^2 T = 0; \qquad X'' + \lambda^2 X = 0 \tag{5}$$

- Because T must be sinusoidal, $\lambda \neq 0$, so the ODEs give the solutions

$$T = C_1 \cos c\lambda t + C_2 \sin c\lambda t; \qquad X = D_1 \cos\lambda x + D_2 \sin\lambda x \tag{6}$$

- Plug the BCs into X to get one constant and the eigenvalue λ^2.
- Write the general solution $u(x,t) = TX$ as a Fourier series.
- Plug in the **two** initial conditions to find the other constants for the complete solution.

3. Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{7}$$

- Plug in the separation Ansatz into the equation to get $\frac{X''}{X} = -\frac{Y''}{Y}$.
- Determine which direction gives us a Sturm–Liouville problem; i.e., which direction has homogeneous BCs. If it is the x direction, let the equation above equal $-\lambda^2$. If it is the y direction, let the equation equal λ^2.
- The Sturm–Liouville problem will give you **sine** and **cosine** solutions and the other ODE will give you **sinh** and **cosh** solutions.
- Solve the Sturm–Liouville problem for $\lambda = 0$ and $\lambda \neq 0$.
- Plug in the homogeneous BCs into the Sturm–Liouville solution to find the constants and eigenvalue.
- Write the general solution $u(x, y) = XY$ as a Fourier series.
- Plug in the nonhomogeneous boundary conditions to find the other constants for the complete solution.

3.1. **Laplace in Polar Coordinates.**

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{8}$$

- There is one boundary at the radius, which gives us a condition on either u or its derivative. However, we need another condition in the r direction and two in ϕ.
  - r direction: $|u(r = 0, \theta)| < \infty$; i.e. things better stay bounded in the center.
  - θ direction: $u(r, \pi) = u(r, -\pi)$ and $u'(r, \pi) = u'(r, -\pi)$.
- Plug in the separation Ansatz to get the ODEs

$$\phi'' + \lambda^2\phi = 0; \qquad r^2\rho'' + r\rho' - \lambda^2\rho = 0 \tag{9}$$

- The ϕ equation is our Sturm–Liouville problem since it has periodic conditions, and because these conditions never change we always get the following

$$\phi'' + \lambda^2\phi = 0 \Rightarrow \phi = A\cos\lambda\theta + B\sin\lambda\theta$$
$$\phi(\pi) = \phi(-\pi) \Rightarrow A\cos\lambda\pi + B\sin\lambda\pi = A\cos\lambda\pi - B\sin\lambda\pi \Rightarrow \lambda = n$$
$$\phi'(\pi) = \phi'(-\pi) \Rightarrow -An\sin n\pi + Bn\cos n\pi = An\sin n\pi + Bn\cos n\pi \checkmark$$

  So, both these conditions give us the same results

$$\lambda = n \Rightarrow \phi = A\cos n\theta + B\sin n\theta \tag{10}$$

- For the ρ equation we will also get the same solution every time

$$n = 0 : \rho = C_1 + C_2\ln r; \qquad |u(r = 0, \theta)| < \infty \Rightarrow C_2 = 0 \Rightarrow \rho = C_1$$
$$n \neq 0 : \rho = C_3 r^{-n} + C_4 r^n; \qquad |u(r = 0, \theta)| < \infty \Rightarrow C_3 = 0 \Rightarrow \rho = C_4 r^n$$

- Now we put this into the general solution, which again, will be the same every time

$$u(r, \theta) = C_1 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta \tag{11}$$

- Using the natural boundary condition at the radius, solve for the constants.