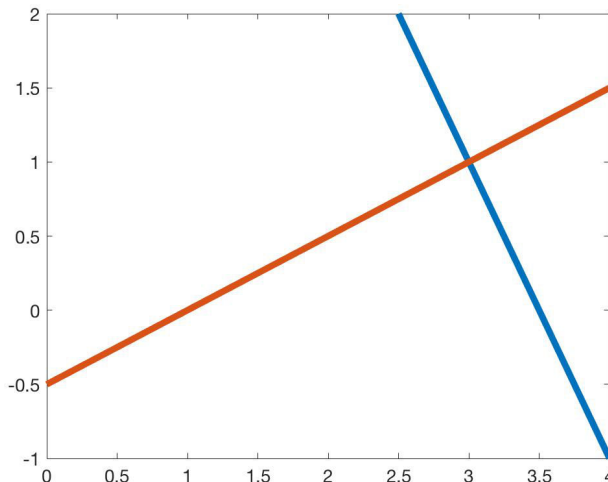


8.1 - 8.6: A CRASH COURSE ON LINEAR ALGEBRA

Linear Algebra is used to solve systems of equations, such as the one below

$$\begin{aligned} x - 2y &= 1 \\ 2x + y &= 7 \end{aligned} \tag{1}$$



Before we can start solving we need to go over some terminology.

Let's first look at some examples of vectors, which you have probably seen before,

- *Scalar Multiplication*:  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$
- *Vector Addition*:  $\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$
- *Linear Combination*:  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$
- *Dot Product* (also known as *Inner Product*):  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1w_1 + v_2w_2$

We also have similar operations for matrices,

- *Matrix Addition* ( $n \times n$ ): Add the corresponding elements,

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{pmatrix}$$

- *Scalar Multiplication*: Multiply the scalar by all of the elements,

$$\gamma A = \gamma \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \gamma a_{11} & \gamma a_{12} & \cdots & \gamma a_{1n} \\ \gamma a_{21} & \gamma a_{22} & \cdots & \gamma a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma a_{n1} & \gamma a_{n2} & \cdots & \gamma a_{nn} \end{pmatrix}$$

- *Matrix Multiplication*: Dot the rows of the first with the columns of the second,

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{in} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \cdots & \sum_{i=1}^n a_{2i}b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{ni}b_{i1} & \sum_{i=1}^n a_{ni}b_{i2} & \cdots & \sum_{i=1}^n a_{ni}b_{in} \end{pmatrix}$$

- *Identity*: We need a matrix that leaves other matrices unchanged when multiplied by it,

$$I = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}$$

- *Transpose*: Switch the rows and columns,

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Notice that  $v^T w = (v_1 \ v_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2 = v \cdot w$ . Furthermore, we can also do  $vw^T$ , which is called the *outer product*. Finally, a matrix  $A$  is called *Symmetric* if  $A = A^T$ .

We also have a few properties of matrix multiplication, transpose, and inverse:

#### Matrix Multiplication Properties.

- Associative:  $(AB)C = A(BC)$
- Distributive:  $A(B + C) = AB + AC$
- Not Commutative:  $AB \neq BA$  in general
- Inverse:  $AA^{-1} = A^{-1}A = I$

#### Transpose Properties.

$$(A^T)^T = A, \quad (A + B)^T = A^T + B^T, \quad (AB)^T = B^T A^T, \quad (kA)^T = kA^T$$

#### Inverse Properties.

$$(A^{-1})^{-1}, \quad (AB)^{-1} = B^{-1}A^{-1}, \quad (A^T)^{-1} = (A^{-1})^T$$

Now we can solve our system of equations (1). First lets do this the middle school way,

$$\begin{array}{r} -2 \times [x - 2y = 1] \\ + [2x + y = 7] \\ \hline 0 + 5y = 5 \end{array} \Rightarrow y = 1 \Rightarrow x = 3.$$

Now lets use Gaussian elimination. You will notice I have changed the notation slightly from class to help you see what's going on. Here  $R_1$  means the first row, and  $R_2$  means the second row. Furthermore, the location of the notation is the row we are operating on. Hopefully this helps any confusion that you may be having.

$$-(2R_1) \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 1 & 7 \end{array} \right) = \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 5 & 5 \end{array} \right) \Rightarrow y = 1 \Rightarrow x = 3.$$

Notice in the last row after the Gaussian elimination our equation becomes  $0x + 5y = 5$ , which is trivial to solve and then using back substitution we can get our  $x$  just as we did for the previous example.

Now lets invert the matrix from the left hand side of our system. In order to do so we append the identity to the **right** of the matrix and our goal is to do Gaussian operations in order to get the left to become the identity, which in turn will transform the **right** into the inverse.

$$\begin{aligned} & -(2R_1) \left( \begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array} \right) = & -(-2R_2/5) \left( \begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array} \right) \\ = & (1/5) \left( \begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 5 & -2 & 1 \end{array} \right) = \left( \begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array} \right) \Rightarrow \left( \begin{array}{cc} 1 & -2 \\ 2 & 1 \end{array} \right)^{-1} = \left( \begin{array}{cc} 1/5 & 2/5 \\ -2/5 & 1/5 \end{array} \right) \end{aligned}$$

Now the natural question becomes: “Is a system of equations always solveable?” The safe answer is, “No”, and here are some counter examples,

Ex: **(Underdetermined)**  $x + y = 1$ ;  $x + y = 2$ . This clearly has no solution since the same quantity cannot equal two different numbers.

Ex: **(Overdetermined)**  $x + y = 1$ ;  $2x + 2y = 2$ . This will have infinitely many solutions because the second equation is just two times the first; i.e. they are the same equation! So we have two unknowns, but only one unique equation, so we will end up getting a line of solutions, which means infinitely many points.

What’s similar between the two coefficient matrices?

$$-(R_1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \quad -(2R_1) \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

For both cases, unlike in (1), the diagonal is not complete; i.e. we do not have *upper triangular* form. Rather, this is called *row-echelon* form. These are called *singular* or equivalently *noninvertible* matrices, whereas (1) is called *nonsingular* or *invertible*.

Before we define these terms formally, let’s first define *linear independence* (L.I.) and *linear dependence* (L.D.).

**Definition 1.** The set of vectors  $\{v_1, v_2, \dots, v_{n-1}, v_n\}$  are said to be linearly independent if  $c_1v_1 + c_2v_2 + \dots + c_{n-1}v_{n-1} + c_nv_n \neq 0$ , where  $c_i$  are scalars, otherwise it is said to be linearly dependent.

**Definition 2.** The expression  $c_1v_1 + c_2v_2 + \dots + c_{n-1}v_{n-1} + c_nv_n$  is said to be a linear combination of  $v_1, v_2, \dots, v_{n-1}, v_n$ .

**Definition 3.** A matrix is said to be invertible if and only if all of its columns are linearly independent.

**Definition 4.** A matrix with  $n$  linearly independent columns is said to have a rank of  $n$ .

Now lets go back to our over and under determined examples to help us on the homework. In order to show a matrix is singular, we need to show a missing “pivot”, which for us will mean a row of zeroes in just the matrix itself before appending anything. This is exactly what we did in the example above. Now, what if we know a matrix is singular and want to determine whether it is over or under determined; i.e. infinitely many solutions or no solution. Then we need to append the right hand side. Let’s go back to our over and under determined examples and do just this.

Ex: **(Underdetermined)** Lets put the system in matrix form (with the right hand side appended) and carry out the Gaussian elimination.

$$-(R_1) \left( \begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array} \right) = \left( \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right)$$

Notice that the bottom row translates to  $0x + 0y = 1$ , and since  $0 \neq 1$ , this system of equations has no solution.

Ex: **(Overdetermined)** Again, as we did above,

$$-(2R_1) \left( \begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array} \right) = \left( \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

Notice that the bottom this time is  $0x + 0y = 0$ , which means there is only one equation and two unknowns, so the system can be solved using an infinite number of ordered pairs.

So for the homework, all you have to do for no solution is show the bottom row is 0 on the left hand side and nonzero on the right hand side. For infinitely many solutions, all you have to do is show the bottom row is 0 for both the right and left hand sides.

Now we move on to determinants. We know that with scalars the *absolute value* is the distance from zero. We can do a similar thing with vectors using either the dot product or Pythagorean theorem (i.e. the distance formula) to give us a *modulus*. With Matrices we have the idea of determinants, which are  $n$ -dimensional volumes. We won’t need to know too much about determinants for this class, but we should know how to compute  $2 \times 2$  and  $3 \times 3$  determinants.

Ex:  $(2 \times 2)$ :  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex:  $(3 \times 3)$ : Here we use a method called expansion by co-factors, however you are free to use any method you are comfortable with.

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$