14.1 and 14.2 PDEs in Polar and Cylindrical Coordinates

We derived the Laplacian in polar coordinates in class, which gave us

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{1}$$

Cylindrical coordinates is just like polar coordinates, except with an added vertical component that doesn't add any complications. So we shall focus on polar coordinates.

Now, consider a disk of radius R. Since we have a second order PDE in two spatial directions, we need two boundary conditions in each direction. The natural boundary condition is $u(r = R, \theta) = f(\theta)$, but we need another one. Notice that in our new Laplacian we have $1/r$ and $1/r^2$, so we have the minor issue of a singularity at the origin. Since we know things don't blow up unless you give it greater and greater energy, we require $|u(r = 0, \theta)| < \infty$. Now, for the θ direction, we have a case similar to the circular rod problem we did. Since this is periodic we require $u(r, \pi) = u(r, -\pi)$ and $u_\theta(r, \pi) = u_\theta(r, -\pi)$.

There is another slight complication for this. In order to solve the PDE we need to know how to solve the Cauchy–Euler equation:

$$x^2 y'' + \alpha x y' + \beta y = 0. \tag{2}$$

Preliminaries: Cauchy–Euler Equation. Consider the ODE

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = 0 \tag{3}$$

This has a singular point because if we put this into standard form we get

$$y'' + \alpha\frac{1}{x}y' + \beta\frac{1}{x^2}y = 0,$$

which violates the existence and uniqueness theorem at $x = 0$. We obviously don't know how to deal with this problem. But there is a similar problem that we do know how to deal with,

$$y''(\xi) + ay'(\xi) + by(\xi) = 0 \tag{4}$$

Basically we need to make a change of variables on x in order to get rid of the $x's$ in the coefficients. What do we know that gives us $1/x$ every time we differentiate? $\xi = \ln x$ does the trick. Taking the derivatives are a little different than what we are used to, but very intuitive due to Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\frac{dy}{d\xi},$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\left(e^{-\xi}\frac{dy}{d\xi}\right)' = \frac{1}{x}\left(-e^{-\xi}\frac{dy}{d\xi} + e^{-\xi}\frac{d^2y}{dx^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right)$$

Plugging this back into (3) gives us

$$\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi} + \alpha\frac{dy}{d\xi} + \beta y = y'' + ay' + by = 0$$

To solve (4) we use the ansatz $y = \exp(r\xi)$, so to solve (3) we use $y = x^r$. Lets think of a slightly more general second order ODE for this part

$$Ax^2y'' + Bxy' + Cy = 0$$

Then plugging into this gives

$$Ax^2[r(r-1)]x^{r-2} + Bxrx^{r-1} + Cx^r = Ar(r-1)x^r + Brx^r + Cx^r = 0 \Rightarrow Ar(r-1) + Br + C = 0.$$

This is our <u>characteristic polynomial</u> of Euler's equation. And we have the usual cases:

Cases	Solution	Comment
Distinct Roots	$y = c_1x^{r_1} + c_2x^{r_2}$	
Repeated Roots	$y = (c_1 + c_2\ln\lvert x\rvert)x^r$	because $\xi = \ln x$
Complex Conjugate Roots	$y = x^\lambda(A\cos(\mu\ln x) + B\sin(\mu\ln x))$	where $r = \lambda \pm i\mu$

Now lets do some problems

Ex: $y'' - y' + y = 0$

Solution: The characteristic polynomial is $r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0$, so we have repeated roots $r = 1$, then $y = (c_1 + c_2 \ln|x|)x;\ x \neq 0$.

Ex: $y'' - 4y' + 4y = 0$

Solution: The characteristic polynomial is $r(r-1) - 4r + 4 = r^2 - 5r + 4 = (r-1)(r-4) = 0$, then $y = c_1 x + c_2 x^4;\ x \neq 0$.

Ex: $y'' + 2y' + 4y = 0$

Solution: The characteristic polynomial is $r(r-1)+2r+4 = r^2+r+4 = 0$, then $r = (-1 \pm i\sqrt{5})/2$, so

$$y = |x|^{-1/2}\left[A\cos\left(\frac{\sqrt{15}}{2}\ln|x|\right) + B\sin\left(\frac{\sqrt{15}}{2}\ln|x|\right)\right].$$

Now, we are equipped to solve Laplace's equation on a disk.

Ex: Consider the steady-state heat conduction problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0; \qquad u(r = R, \theta) = f(\theta) \tag{5}$$

Solution: Let $u(r,\theta) = \varphi(\theta)\rho(r)$. Plugging into the PDE gives us

$$\varphi\rho'' + \frac{1}{r}\varphi\rho' + \frac{1}{r^2}\varphi''\rho = 0 \Rightarrow \left(r^2\rho'' + r\rho'\right)\frac{1}{\rho} = -\frac{\varphi''}{\varphi}.$$

Notice that our Sturm–Liouville problem would be in the θ direction since we don't have homogeneous boundary conditions in r, but we do have periodic boundary conditions in θ, which behave similarly to homogeneous boundary conditions as we saw with the circular rod problem. So, we let the RHS be λ^2; i.e.

$$\left(r^2\rho'' + r\rho'\right)\frac{1}{\rho} = -\frac{\varphi''}{\varphi} = \lambda^2$$

This gives us two ODEs with the corresponding boundary conditions

$$\begin{aligned} &\varphi'' + \lambda^2\varphi = 0; \qquad \varphi(\pi) = \varphi(-\pi), \quad \varphi'(\pi) = \varphi'(-\pi) \\ &r^2\rho'' + r\rho - \lambda^2\rho = 0; \qquad |\rho(0)| < \infty \end{aligned} \tag{6}$$

Notice that we leave out the outer boundary condition (the only prescribed condition) since we need the full equation to satisfy it because it is nonhomogeneous, nonperiodic, and not a bound.

Now, since θ is periodic, it must either be sinusoidal or a constant; i.e., it can't be linear or exponential (no sinh and/or cosh). So we get

$$\varphi = C_1\cos(\lambda\theta) + C_2\sin(\lambda\theta)$$

Invoking the periodic boundary condition gives us

$$\varphi(-\pi) = \varphi(\pi) \Rightarrow C_1\cos(\lambda\pi) - C_2\sin(\lambda\pi) = C_1\cos(\lambda\pi) + C_2\sin(\lambda\pi) \Rightarrow \sin(\lambda\pi) = 0 \Rightarrow \lambda = n.$$

We can verify the same result for the derivative.

Now we solve the ρ equation. For $n = 0$ we have

$$r^2\rho'' + r\rho' = 0 \Rightarrow \mu(\mu - 1) + \mu = \mu^2 = 0 \Rightarrow \rho = D_1 + D_2\ln r$$

Notice that we could also solve this ODE via separation of variables, but this way is less time consuming. Since $|u(0,\theta)| < \infty$, $D_2 = 0 \Rightarrow \rho = D_1$.

Now, we look at $n \neq 0$,

$$r^2\rho'' + r\rho' - n^2\rho = 0 \Rightarrow \mu(\mu-1) + \mu - n^2 = \mu^2 - n^2 = 0 \Rightarrow \mu = \pm n \Rightarrow \rho = D_2 r^n + D_3 r^{-n}.$$

Since $|u(0,\theta)| < \infty$, $D_3 = 0$, then $\rho = D_2 r^n$. Therefore, the general solution is

$$u(r,\theta) = D_1 + \sum_{n=1}^{\infty} A_n r^n\cos(n\theta) + B_n r^n\sin(n\theta). \tag{7}$$

Now we must satisfy the boundary condition

$$u(R,\theta) = D_1 + \sum_{n=1}^{\infty} A_n R^n \cos(n\theta) + B_n R^n \sin(n\theta) = f(\theta)$$

This is just like our Fourier series, so in general we get the coefficients by doing the following integrals

$$\begin{aligned} D_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \\ A_n R^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta, \\ B_n R^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta \end{aligned}$$