8.8: The eigenvalue problem

There is a special type of problem called the *eigenvalue problem*, which is an $Ax = b$ type problem, but now $Ax = \lambda x$ where λ is some value that admits nontrivial values of x. In order to solve the problem we need to isolate x, and since our dimensions need to be consistent we arrive at

$$(A - \lambda I)x = 0 \Rightarrow \det(A = \lambda I) = 0 \tag{1}$$

because the determinant is the measure of the "size" of the matrix. If we want nontrivial values for x, then the size of $A - \lambda I$ must be zero. Once we solve for the λ's we can solve for the respective values of x. Here λ is called an *eigenvalue* and x is called an eigenvector.

Ex: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution: The eigenvalues are,

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 1$$

And the associated eigenvectors are

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

12) The eigenvalues are,

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 + 1 = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Then the eigenvectors are,

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}; \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} i \\ -1 \end{pmatrix}$$

Notice that these two eigenvectors are complex conjugates since the eigenvalues are complex conjugates. So it is only necessary to compute one eigenvector and we get the other one for free!

18) The eigenvalues are,

$$\begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(4 - 4\lambda + \lambda^2 - 1) = (1-\lambda)(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 3.$$

Then the eigenvectors are,

$$\begin{pmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} x = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x_2,\ x_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Homework Tip. Lets suppose we have some factored polynomial: $(x+2)(x-1)(x-3)$, and we want to figure out for what intervals $(x+2)(x-1)(x-3) = 0$, $(x+2)(x-1)(x-3) < 0$, or $(x+2)(x-1)(x-3) > 0$. The zeroes are the easiest since they are just the roots of the solution, so we can plot $x = -2, 1, 3$ on a number line and test a point in each interval $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, $(3, \infty)$ to see if the polynomial is positive or negative.

10.1: Linear Systems of ODEs

We study differential equations because real world processes are governed by rates. You have all seen first order and even higher order ODEs representing one process, but what if there are multiple coupled processes? Then we need an array of ODEs,

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ \dot{x}_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

This can be written as the matrix equation,

$$\frac{d}{dt}\overrightarrow{x} = A(t)\overrightarrow{x} + \overrightarrow{f}(t) \tag{2}$$

Then the solution to the ODE involves two parts, a characteristic part that solves the homogeneous equation, and a particular part that supplements the solution to match the forcing on the right hand side. That is, $\overrightarrow{x} = \overrightarrow{x}_c + \overrightarrow{x}_p$. Here $\overrightarrow{x}_c = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)} + \cdots + c_n\overrightarrow{x}^{(n)}$; i.e., it is a superposition of n solutions since we have n ODEs, and this solves the homogeneous equation

$$\frac{d}{dt}\overrightarrow{x} = A(t)\overrightarrow{x} \tag{3}$$

Now lets go over some definitions that you may have seen before, but it's worth revising.

Definition 1. The set of functions $\{h_1, h_2, \ldots, h_{n-1}, h_n\}$ are said to be linearly independent if $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n \neq 0$, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n$ is said to be a linear combination of $h_1, h_2, \ldots, h_{n-1}, h_n$.

Theorem 1. *If $\overrightarrow{x}^{(1)}, \overrightarrow{x}^{(2)}, \ldots, \overrightarrow{x}^{(n-1)}, \overrightarrow{x}^{(n)}$ are solutions to* (3)*, then any linear combination of $\overrightarrow{x}$'s are also solutions.*

For example, $\overrightarrow{x} = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)}$, $y = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)} + \cdots + c_n\overrightarrow{x}^{(n)}$, etc. are also solutions.

Now we define what the Wronskian is, which you have seen before, but in a slightly different context.

Definition 3. The *Wronskian* is

$$W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n)} \\ \cdots & \cdots & \vdots & \cdots \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} \end{vmatrix} \tag{4}$$

Theorem 2. *Suppose $\overrightarrow{x}^{(1)}, \overrightarrow{x}^{(2)}, \ldots, \overrightarrow{x}^{(n-1)}, \overrightarrow{x}^{(n)}$ are solutions to* (3) *on an interval I, with the usual initial conditions, then $W \neq 0$ guarantees they are linearly independent on $\in I$.*

So the rewording of the above theorem implies that if the Wronskian is zero at a single point then the function may still be linearly independent.

Definition 4. The set of all linearly independent solutions of an ODE is called the fundamental set of solutions for that ODE.