10.2: Homogeneous Linear Systems

Consider the 2×2 ODE $\frac{d\vec{x}}{dt} = A(t)\vec{x}$. As we have seen with the linear algebra problems, a 2-dimensional eigenvalue problem has three types of solutions:

Case	Eigenvalue	Eigenvector	General Solution
Real Distinct	$A\vec{v} = \lambda\vec{v} \Rightarrow \lambda = \lambda_1,\ \lambda_2$	$\vec{v} = \vec{v}_1,\ \vec{v}_2$	$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$
Real Repeated	$A\vec{v} = \lambda\vec{v} \Rightarrow \lambda = \lambda$	$\vec{v} = \vec{v},\ A\vec{w} = \vec{v}$	$\vec{x} = c_1\vec{v}e^{\lambda t} + c_2 e^{\lambda t}[\vec{v}t + \vec{w}]$
Complex Conjugate	$\lambda = \xi \pm i\theta$	$\vec{v} = \vec{\nu} \pm i\vec{\omega}$	$\vec{x} = c_1 e^{\xi t}[\vec{\nu}\sin\theta t + \omega\cos\theta t]$ $+ c_2 e^{\xi t}[\vec{\nu}\cos\theta t - \vec{\omega}\cos\theta t]$

Now, lets go over a bunch of examples

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\vec{x}$

Solution: First find the eigenvalues,

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 6 - \lambda + \lambda^2 + 4 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = -1, 2$$

Then we find the eigenvectors,

$$\begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}\vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}.$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\vec{x}$

Solution: The eigenvalues are

$$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = \lambda^2 - 9 + 9 = \lambda^2 = 0 \Rightarrow \lambda = 0.$$

The eigenvector is,

$$\begin{pmatrix} 3 & 1 \\ 9 & -3 \end{pmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

And the generalized eigenvector is

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\vec{w} = \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]$$

Ex: Consider the system of ODEs $\dot{x} = 2x - 5y$, $\dot{y} = x - 2y$.

Solution: This translates into the matrix ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x}$. We take the eigenvalues as usual,

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (4-\lambda^2) + 5 = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Then the eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}; \qquad \vec{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

Then our solution is

$$\hat{x} = c_1 \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-it} = c_1 \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) + c_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} (\cos t - i \sin t)$$

$$= c_1 \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \right] + c_2 \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t - 2\sin t \\ -\sin t \end{pmatrix} \right]$$

So, our real solution would be,

$$x = (c_1 + c_2) \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + (c_1 - c_2) \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Notice, since the eigenvectors are complex conjugates, we only need one eigenvector to find our solution. This is what we will do from now on. **Don't do the problem the way I showed the instructional example!**

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \vec{x}$; $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution: Here we will solve an IVP. We take the eigenvalues,

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = -3 + 2\lambda + \lambda^2 + 5 = \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) = -1 \pm i.$$

The eigenvectors for $\lambda = -1 + i$ is,

$$x^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \Rightarrow \hat{x} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i \sin t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Then our general solution is,

$$x = Ae^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + Be^{-t} \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Now we plug in our initial conditions,

$$x(0) = A \begin{pmatrix} 2 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow A = 1 \Rightarrow B = -1$$

Then our solution is,

$$x = e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$; $\vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solution: Again we find the eigenvalues

$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) = 0 \Rightarrow \lambda = 2,\ 4.$$

Then we find the eigenvectors

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Our general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

Then we solve for the constants

$$\vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow c_1 = -3/2 \Rightarrow c_2 = 7/2$$

Then the full solution is

$$\vec{x} = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \vec{x}$

Solution: The eigenvalues are

$$\begin{vmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2 = 0 \Rightarrow \lambda = 6.$$

Then the eigenvector and the generalized eigenvector are

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \qquad \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{w} = \vec{v} \Rightarrow \vec{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 e^{6t} \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Additional Examples. Here are some additional examples for **distinct real eigenvalues** that we didn't do in class. You must understand how to do these problems before you can hope to solve the the distinct real and complex conjugate cases, so if you are having trouble please go through these examples step by step.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and comment on what happens as $t \to \infty$ for $c_2 = 0$ and $c_2 \neq 0$.

Solutions: Again we find the eigenvalues,

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

The eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Here if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$ and comment on what happens as $t \to \infty$.

Solution: Again,

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = (\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = -1, -3.$$

And the eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Our solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

Here $x \to 0$.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \vec{x}$ and comment on what happens as $t \to \infty$ for $c_2 = 0$ and $c_2 \neq 0$.

Solutions: Again the eigenvalues are

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 6 = -6 - \lambda + \lambda^2 + 6 = \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0, 1$$

with the eigenvectors,

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Then our solution is

$$x = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

So our solution behaves as follows: if $c_2 = 0$, $x = c_1(-2, 1)$; i.e. the first eigenvector. If $c_2 \neq 0$, $x \to \infty$.