12.2: Fourier Series

This definition allows us to construct a space of functions out of two simple functions. Now equipped with our new machinery we can derive a series representation that is ideal for periodic functions. We did this in class, but here I shall just remind you of the formulas:

Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right];$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right),\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

Now lets do some problems. While a lot of these want plotting, we did them in class, so I won't show them here, but make sure you know how to plot these things.

Ex: Find the Fourier Series of the function

$$f(x) = \begin{cases} 1 & -L < x < 0, \\ 0 & 0 \le x < L; \end{cases}$$

(a) Sketch it!
(b) We first do a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{0} dx = 1.$$

Notice that we always do a_0 separately. Then we do a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\cos\left(\frac{n\pi x}{L}\right)dx = \cancel{\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)}\Bigg|_{-L}^{0} \longrightarrow 0$$

Finally, for b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0}$$
$$= -\frac{1}{n\pi} + \frac{1}{n\pi}\cos(n\pi) = \frac{-1 + (-1)^n}{n\pi} = -\frac{2}{n\pi}\begin{cases} 1 & n\text{odd, i.e. } n = 2k+1;\ k = 0, \pm 1, \pm 2, \dots \\ 0 & n\text{even, i.e. } n = 2k;\ k = 0, \pm 1, \pm 2, \dots \end{cases}$$

Then our Fourier series becomes

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{1}{L}(2k+1)\pi x\right).$$

Ex: Find the Fourier Series of the function $f(x) = x^2/2$ on $[-2, 2]$

(a) Plot it!
(b) Again, we do a_0 first

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}dx = \frac{x^3}{12}\Bigg|_{-2}^{2} = \frac{4}{3}.$$

Now to do a_n we need to do by parts twice, which you can do yourselves. I'll just give the final form of the antiderivative.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 \frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx$$
$$= \left[\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{(n\pi)^3}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 = \frac{8}{(n\pi)^2}\cos(n\pi) = (-1)^n\frac{8}{(n\pi)^2}.$$

For b_n we get

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\sin\left(\frac{n\pi x}{L}\right)dx = 0.$$

because we are integrating an odd function on a symmetric interval. Then our Fourier series is

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

15) This is a book problem.

First we find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}e^x\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^p i - e^{-\pi}\right) = \frac{2}{\pi}\sinh\pi$$

Then we find a_n via "by parts" using $u = \cos nx \Rightarrow du = -n\sin nx dx$ and $dv = e^x dx \Rightarrow v = e^x$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx = \frac{1}{\pi}\left[e^x\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x \sin x dx\right]$$

Then we do another by parts: $u = \sin nx \Rightarrow du = n\cos nx$ and $dv = e^x dx \Rightarrow v = e^x$

$$\frac{1}{\pi}\left\{e^x\cos nx\Big|_{-\pi}^{\pi} + n\left[\cancelto{0}{e^x\sin nx\Big|_{-\pi}^{\pi}} - n\int_{-\pi}^{\pi} e^x\cos nx dx\right]\right\}$$
$$= \frac{1}{\pi}\left\{\left(e^{\pi} - e^{-\pi}\right)(-1)^n - n^2\int_{-\pi}^{\pi} e^x\cos nx dx\right\} = (-1)^n\frac{2}{\pi}\sinh\pi - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx$$

Now we notice that we have $\int_{-\pi}^{\pi} e^x\cos nx dx$ on both the right and left hand sides, so we can combine them,

$$\frac{n^2+1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx = (-1)^n\frac{2}{\pi}\sinh\pi \Rightarrow a_n = \frac{(-1)^n}{n^2+1}\cdot\frac{2}{\pi}\sinh\pi$$

For b_n we have something similar so I will skip a bunch of steps,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx dx = \frac{1}{\pi}\left\{-e^x\sin nx\Big|_{-\pi}^{\pi} - n\left[e^x\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x\sin nx dx\right]\right\}$$
$$\Rightarrow \frac{n^2+1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx dx = -(-1)^n\frac{2n}{\pi}\sinh\pi \Rightarrow b_n = -\frac{(-1)^n}{n^2+1}\cdot\frac{2n}{\pi}\sinh\pi$$

Then the Fourier Series is

$$f(x) = \frac{2}{\pi}\sinh\pi\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right].$$

12.3: Even and Odd Functions

As we saw for the last problem in the preceding section, it can be useful to know whether or not a function is odd or even. Also, many times we will want the Fourier series of a non-periodic function. In order to do this we need to create a periodic function that includes our non-periodic function. Instead of creating something that is neither odd nor even if we create an even or odd function we can save a lot of time. Before we see these techniques lets define some terms and develop the theory.

Definition 1. Consider the function $f(x)$ such that $f(-x) = f(x)$, then f is said to be even.

Definition 2. Consider a function $f(x)$ such that $f(-x) = -f(x)$, then f is said to be odd.

There are some important properties that we should keep in mind.

Properties.

- Sum/difference of two even functions is even.
- Sum/difference of two odd functions is odd.
- Sum/difference of an even and an odd function is neither even nor odd.
- Product/quotient of two even functions is even.
- Product/quotient of two odd functions is even.
- Product/quotient of an even function and an odd function is odd.
- If f is even, $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.
- If f is odd, $\int_{-L}^{L} f(x)dx = 0$.

Now we can think of a Fourier cosine series and Fourier sine series. These can be derived straight from the Fourier series equations so it's best not to memorize these formulas.

Fourier cosine series. If f is an even periodic function generated on $-L \leq x \leq L$, then $b_n = 0$, so

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{2}$$

Fourier sine series. If f is an odd periodic function generated on $-L \leq x \leq L$, then $a_n = 0$, so

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{3}$$

For the next few problems we just apply the definition of odd and even functions.

(1) Odd 5) Even 6) Neither

Periodic Extensions. Suppose a function f is defined only on $[0, L]$. If we want to find the Fourier series of this we need to make a periodic function that "includes" f. These are called periodic extensions and can either be odd or even.

For these problems we did the sketching in class. Here I will do the problems that requires calculations

Ex: Find the Fourier Sine Series of $f(x) = L - x$ on $[0, L]$.

(a) Notice that for odd extensions our periodic function of period $2L$ becomes

$$g(x) = \begin{cases} -f(-x) & -L < x < 0, \\ f(x) & 0 < x < L; \end{cases} = \begin{cases} -L - x & -L < x < 0, \\ L - x & 0 < x < L; \end{cases}$$

We know that for odd extensions we'll get a sine series so we only do the sine calculations,

$$b_n = \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)dx = -(L-x)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{n\pi} + \cancel{\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L} \to 0$$

Then our Fourier sine series is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

(b) Sketch the solution for $L = 4$.

Ex: Find the Fourier Sine and Cosine series of the following function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 0 & \text{for } 1 < x < 2 \end{cases}$$

(a) Sketch the even and odd extensions of the function.

(b) For the cosine series we have

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \int_0^1 x dx = \frac{1}{2}.$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1$$
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}.$$

Notice that for this problem we can't simplify the indices in any reasonable manner, so we leave it as is. So the Fourier cosine series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\right)\cos\left(\frac{n\pi x}{2}\right).$$

Now, for the sine series we have

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$
$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

Then our Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)$$