

13.3 HEAT EQUATION EXAMPLES

Consider the heat equation with a generic initial condition,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = f(x). \tag{1}$$

with the following boundary conditions

Ex:  $u(0, t) = u(L, t) = 0$ .

**Solution:** We make the Ansatz,  $u(x, t) = T(t)X(x)$ . Then we plug this into our heat equation

$$u_t = T'(t)X(x), \quad u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}.$$

Since the LHS is a function of  $t$  alone, and the RHS is a function of  $x$  alone, and since they are equal, they must equal a constant. Lets call it  $-\lambda^2$ . Then we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \tag{2}$$

Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue  $\lambda^2$ . Now we must solve the two differential equations.

The  $T$  equation is the easiest to solve

$$\frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \frac{dT}{dt} = -k\lambda^2 T \Rightarrow \frac{dT}{T} = -k\lambda^2 dt \Rightarrow \int \frac{dT}{T} = \int -k\lambda^2 dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t}$$

Notice that we don't include the constant in front of the exponential, and that is because the  $X$  equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don't have to though.

Now, we solve the  $X$  equation by recalling our Sturm-Liouville problems

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A \cos \lambda x + B \sin \lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0.$$

If we look at the  $\lambda = 0$  case we have  $X(0) = c_2 = 0$  and  $X(L) = Lc_1 = 0$ , so  $X \equiv 0$ .

Now we look at the  $\lambda \neq 0$  case.  $X(0) = A = 0$  and

$$X(L) = B \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin \frac{n\pi}{L} x \text{ and } T_n = e^{-k(\frac{n\pi}{L})^2 t}$$

Next we combine the  $T$  and  $X$  solutions to get the general solutions,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} \tag{3}$$

And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have

$$u(x, 0) = \sum_{n=1} B_n \sin \frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Then our full solution is

$$u(x, t) = \frac{2}{L} \sum_{n=1} \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{4}$$

Ex:  $u_x(0, t) = u_x(L, t) = 0$ .

**Solution:** We know from the first example that  $T = e^{-k\lambda^2 t}$ .

For the  $X$  equation we need to look at our two cases. For  $\lambda = 0$  we have  $X = c_1x + c_2$ , and  $X'(x) = c_1$ , so for both boundaries  $X'(0) = c_1 = X'(L)$ . These leaves us with a constant  $X = c_2$ .

For the  $\lambda \neq 0$  case we have

$$X = A \cos \lambda x + B \sin \lambda x \Rightarrow X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x$$

Then we get  $X'(0) = \lambda B = 0$  and

$$X'(L) = -\lambda A \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = A_n \cos \frac{n\pi x}{L} \text{ and } T_n = e^{-k(\frac{n\pi}{L})^2 t}$$

Next we combine the  $T$  and  $X$  solutions to get our general solution

$$u(x, t) = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} \quad (5)$$

Now we find our coefficients by invoking the initial condition and using Fourier Series

$$u(x, 0) = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x)$$

This gives us

$$c_2 = \frac{1}{L} \int_0^L f(x) dx$$

and

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Combining everything we get the full solution

$$u(x, t) = \frac{1}{L} \int_0^L f(x) dx + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad (6)$$

Ex: Now lets think of heat transfer in a circle. If we go around in one direction we hit  $x = -L$  and in the other direction  $x = L$ , but these are the same point. So we get the following boundary conditions

$$u(-L, t) = u(L, t), u_x(-L, t) = u_x(L, t) \quad (7)$$

**Solution:** We know from the previous two problems that our solutions will be

$$T = e^{-k\lambda^2 t}$$

$$X = c_1x + c_2 \text{ for } \lambda = 0$$

$$X = A \cos \lambda x + B \sin \lambda x \text{ for } \lambda \neq 0$$

For  $\lambda = 0$ ,  $X(L) = c_1L + c_2$  and  $X(-L) = -c_1L + c_2$ , so  $c_1 = 0$ . And the derivative is trivially satisfied.

For  $\lambda \neq 0$ ,

$$X(L) = X(-L) \Rightarrow A \cos \lambda L + B \sin \lambda L = A \cos \lambda L - B \sin \lambda L \Rightarrow \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

And

$$X'(L) = X'(-L) \Rightarrow -\lambda A \sin \lambda L + \lambda B \sin \lambda L = \lambda A \sin \lambda L + \lambda B \cos \lambda L \Rightarrow \sin \lambda L = 0$$

But we already showed this. So, we need to keep both coefficients. Then our solution for  $X$ , which as we saw in previous conditions (for the heat equation) is just the initial condition of the general solution, is

$$X = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} = u(x, 0) = f(x) \quad (8)$$

Now we use Fourier series to solve for the coefficients,

$$c_2 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Putting everything back into the general solution gives us

$$u(x, t) = \frac{1}{L} \int_0^L f(x) dx + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx + \sin \frac{n\pi x}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad (9)$$