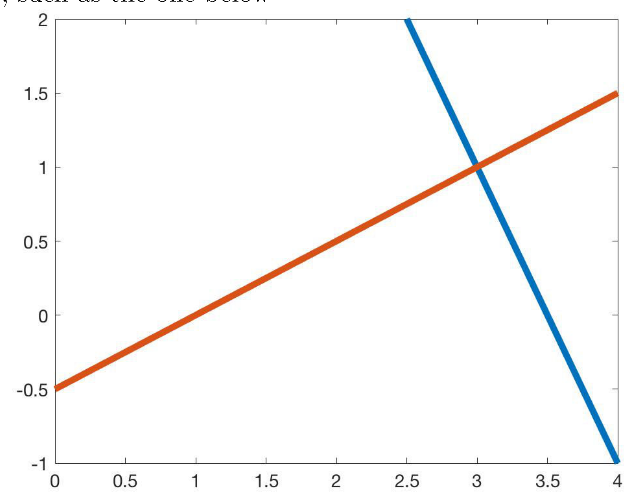
8.1 - 8.6: A crash course on Linear Algebra

Linear Algebra is used to solve systems of equations, such as the one below

$$\begin{aligned} {\color{red} x - 2y = 1} \\ {\color{blue} 2x + y = 7} \end{aligned} \qquad (1)$$



Before we can start solving we need to go over some terminology.

Let's first look at some examples of vectors, which you have probably seen before,

- *Scalar Multiplication*: $3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$
- *Vector Addition*: $\begin{bmatrix}3\\6\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$
- *Linear Combination*: $3\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$
- *Dot Product* (also known as *Inner Product*): $\begin{pmatrix}v_1\\v_2\end{pmatrix} \cdot \begin{pmatrix}w_1\\w_2\end{pmatrix} = v_1w_1 + v_2w_2$

We also have similar operations for matrices,

- *Matrix Addition* ($n \times n$): Add the corresponding elements,

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{pmatrix}$$

- *Scalar Multiplication*: Multiply the scalar by all of the elements,

$$\gamma A = \gamma \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \gamma a_{11} & \gamma a_{12} & \cdots & \gamma a_{1n} \\ \gamma a_{21} & \gamma a_{22} & \cdots & \gamma a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma a_{n1} & \gamma a_{n2} & \cdots & \gamma a_{nn} \end{pmatrix}$$

- *Matrix Multiplication*: Dot the rows of the first with the columns of the second,

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{in} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{21} & \cdots & \sum_{i=1}^n a_{2i}b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{ni}b_{i1} & \sum_{i=1}^n a_{ni}b_{i2} & \cdots & \sum_{i=1}^n a_{ni}b_{in} \end{pmatrix}$$

- *Identity*: We need a matrix that leaves other matrices unchanged when multiplied by it,

$$I = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}$$

- *Transpose*: Switch the rows and columns,

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

  Notice that $v^T w = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2 = v \cdot w$. Furthermore, we can also do vw^T, which is called the *outer product*. Finally, a matrix A is called *Symmetric* if $A = A^T$.

We also have a few properties of matrix multiplication, transpose, and inverse:

Matrix Multiplication Properties.

- Associative: $(AB)C = A(BC)$
- Distributive: $A(B + C) = AB + AC$
- Not Commutative: $AB \neq BA$ in general
- Inverse: $AA^{-1} = A^{-1}A = I$

Transpose Properties.

$$\left(A^T\right)^T = A, \qquad (A + B)^T = A^T + B^T, \qquad (AB)^T = B^T A^T, \qquad (kA)^T = kA^T$$

Inverse Properties.

$$\left(A^{-1}\right)^{-1}, \qquad (AB)^{-1} = B^{-1}A^{-1}, \qquad \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Now we can solve our system of equations (**??**). First lets do this the middle school way,

$$\begin{array}{r} -2 \times [x - 2y = 1] \\ + \quad [2x + y = 7] \\ \hline 0 + 5y = 5 \end{array} \qquad \Rightarrow y = 1 \Rightarrow x = 3.$$

Now lets use Gaussian elimination. You will notice I have changed the notation slightly from class to help you see what's going on. Here R_1 means the first row, and R_2 means the second row. Furthermore, the location of the notation is the row we are operating on. Hopefully this helps any confusion that you may be having.

$$\begin{matrix} \\ -(2R_1) \end{matrix} \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 1 & 7 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 5 & 5 \end{array}\right) \Rightarrow y = 1 \Rightarrow x = 3.$$

Notice in the last row after the Gaussian elimination our equation becomes $0x + 5y = 5$, which is trivial to solve and then using back substitution we can get our x just as we did for the previous example.

Now lets invert the matrix from the left hand side of our system. In order to do so we append the identity to the right of the matrix and our goal is to do Gaussian operations in order to get the left to become the identity, which in turn will transform the right into the inverse.

$$\begin{matrix} \\ -(2R_1) \end{matrix} \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) = \begin{matrix} -(-2R_2/5) \\ \\ \end{matrix} \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right)$$

$$= \begin{matrix} \\ (1/5) \end{matrix} \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 5 & -2 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix}$$