13.3 Heat Equation Examples

Consider the heat equation with a generic initial condition,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = f(x). \tag{1}$$

with the following boundary conditions

Ex: $u(0,t) = u(L,t) = 0$.

Solution: We make the Ansatz, $u(x,t) = T(t)X(x)$. Then we plug this into our heat equation

$$u_t = T'(t)X(x),\ u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}.$$

Since the LHS is a function of t alone, and the RHS is a function of x alone, and since they are equal, they must equal a constant. Lets call it $-\lambda^2$. Then we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \tag{2}$$

Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue λ^2. Now we must solve the two differential equations.

The T equation is the easiest to solve

$$\frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \frac{dT}{dt} = -k\lambda^2 T \Rightarrow \frac{dT}{T} = -k\lambda^2 dt \Rightarrow \int \frac{dT}{T} = \int -k\lambda^2 dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t}$$

Notice that we don't include the constant in front of the exponential, and that is because the X equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don't have to though.

Now, we solve the X equation by recalling our Sturm-Liouville problems

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0.$$

If we look at the $\lambda = 0$ case we have $X(0) = c_2 = 0$ and $X(L) = Lc_1 = 0$, so $X \equiv 0$.

Now we look at the $\lambda \neq 0$ case. $X(0) = A = 0$ and

$$X(L) = X(L) = B\sin\lambda x = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin\frac{n\pi}{L}x \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Next we combine the T and X solutions to get the general solutions,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{3}$$

And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Then our full solution is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{4}$$

Ex: $u_x(0,t) = u_x(L,t) = 0$.

Solution: We know from the first example that $T = e^{-k\lambda^2 t}$.

For the X equation we need to look at our two cases. For $\lambda = 0$ we have $X = c_1 x + c_2$, and $X'(x) = c_1$, so for both boundaries $X'(0) = c_1 = X'(L)$. These leaves us with a constant $X = c_2$.

For the $\lambda \neq 0$ case we have

$$X = A\cos\lambda x + B\sin\lambda x \Rightarrow X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$$

Then we get $X'(0) = \lambda B = 0$ and

$$X'(L) = -\lambda A\sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = A_n\cos\frac{n\pi x}{L} \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Next we combine the T and X solutions to get our general solution

$$u(x,t) = c_2 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{5}$$

Now we find our coefficients by invoking the initial condition and using Fourier Series

$$u(x,0) = c_2 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} = f(x)$$

This gives us

$$c_2 = \frac{1}{L}\int_0^L f(x)dx$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Combining everything we get the full solution

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{6}$$

Ex: Now lets think of heat transfer in a circle. If we go around in one direction we hit $x = -L$ and in the other direction $x = L$, but these are the same point. So we get the following boundary conditions

$$u(-L,t) = u(L,t), u_x(-L,t) = u_x(L,t) \tag{7}$$

Solution: We know from the previous two problems that our solutions will be

$$T = e^{-k\lambda^2 t}$$
$$X = c_1 x + c_2 \text{ for } \lambda = 0$$
$$X = A\cos\lambda x + B\sin\lambda x \text{ for } \lambda \neq 0$$

For $\lambda = 0$, $X(L) = c_1 L + c_2$ and $X(-L) = -c_1 L + c_2$, so $c_1 = 0$. And the derivative is trivially satisfied.

For $\lambda \neq 0$,

$$X(L) = X(-L) \Rightarrow A\cos\lambda L + B\sin\lambda L = A\cos\lambda L - B\sin\lambda L \Rightarrow \sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

And

$$X'(L) = X'(-L) \Rightarrow -\lambda A\sin\lambda L + \lambda B\cos\lambda L = \lambda A\sin\lambda L + \lambda B\cos\lambda L \Rightarrow \sin\lambda L = 0$$

But we already showed this. So, we need to keep both coefficients. Then our solution for X, which as we saw in previous conditions (for the heat equation) is just the initial condition of the general solution, is

$$X = c_2 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} = u(x,0) = f(x) \tag{8}$$

Now we use Fourier series to solve for the coefficients,

$$c_2 = \frac{1}{L}\int_0^L f(x)dx$$
$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$
$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Putting everything back into the general solution gives us

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^L f(x)\cos\frac{n\pi x}{L}dx + \sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{9}$$