8.6: A crash course on Linear Algebra (continued)

Now the natural question becomes: "Is a system of equations always solveable?" The safe answer is, "No", and here are some counter examples,

Ex: **(Underdetermined)** $x + y = 1$; $x + y = 2$. This clearly has no solution since the same quantity cannot equal two different numbers.

Ex: **(Overdetermined)** $x + y = 1$; $2x + 2y = 2$. This will have infinitely many solutions because the second equation is just two times the first; i.e. they are the same equation! So we have two unknowns, but only one unique equation, so we will end up getting a line of solutions, which means infinitely many points.

What's similar between the two coefficient matrices?

$$\begin{matrix} \\ -(R_1) \end{matrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \begin{matrix} \\ -(2R_1) \end{matrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

For both cases, unlike in the previous week, the diagonal is not complete; i.e. we do not have *upper triangular* form. Rather, this is called *row-echelon* form. These are called *singular* or equivalently *noninvertible* matrices, whereas the example in the previous week is called *nonsingular* or *invertible*.

Before we define these terms formally, let's first define *linear independence* (L.I.) and *linear dependence* (L.D.).

Definition 1. The set of vectors $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ are said to be linearly independent if $c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1} + c_nv_n \neq 0$, where c_i are scalars, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1} + c_nv_n$ is said to be a linear combination of $v_1, v_2, \ldots, v_{n-1}, v_n$.

Definition 3. A matrix is said to be invertible if and only if all of its columns are linearly independent.

Definition 4. A matrix with n linearly independent columns is said to have a rank of n.

Now lets go back to our over and under determined examples to help us on the homework. In order to show a matrix is singular, we need to show a missing "pivot", which for us will mean a row of zeroes in just the matrix itself before appending anything. This is exactly what we did in the example above. Now, what if we know a matrix is singular and want to determine whether it is over or under determined; i.e. infinitely many solutions or no solution. Then we need to append the right hand side. Let's go back to our over and under determined examples and do just this.

Ex: **(Underdetermined)** Lets put the system in matrix form (with the right hand side appended) and carry out the Gaussian elimination.

$$\begin{matrix} \\ -(R_1) \end{matrix}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

Notice that the bottom row translates to $0x + 0y = 1$, and since $0 \neq 1$, this system of equations has no solution.

Ex: **(Overdetermined)** Again, as we did above,

$$\begin{matrix} \\ -(2R_1) \end{matrix}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Notice that the bottom this time is $0x + 0y = 0$, which means there is only one equation and two unknowns, so the system can be solved using an infinite number of ordered pairs.

So for the homework, all you have to do for no solution is show the bottom row is 0 on the left hand side and nonzero on the right hand side. For infinitely many solutions, all you have to do is show the bottom row is 0 for both the right and left hand sides.

Now we move on to determinants. We know that with scalars the *absolute value* is the distance from zero. We can do a similar thing with vectors using either the dot product or Pythagorean theorem (i.e. the distance formula) to give us a *modulus*. With Matrices we have the idea of determinants, which are n-dimensional volumes. We won't need to know too much about determinants for this class, but we should know how to compute 2×2 and 3×3 determinants.

Ex: (2×2): $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex: (3×3): Here we use a method called expansion by co-factors, however you are free to use any method you are comfortable with.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

8.8: The eigenvalue problem

There is a special type of problem called the *eigenvalue problem*, which is an $Ax = b$ type problem, but now $Ax = \lambda x$ where λ is some value that admits nontrivial values of x. In order to solve the problem we need to isolate x, and since our dimensions need to be consistent we arrive at

$$(A - \lambda I)x = 0 \Rightarrow \det(A = \lambda I) = 0 \tag{1}$$

because the determinant is the measure of the "size" of the matrix. If we want nontrivial values for x, then the size of $A - \lambda I$ must be zero. Once we solve for the λ's we can solve for the respective values of x. Here λ is called an *eigenvalue* and x is called an eigenvector.

Ex: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution: The eigenvalues are,

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 1$$

And the associated eigenvectors are

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

12) The eigenvalues are,

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 + 1 = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Then the eigenvectors are,

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}; \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} i \\ -1 \end{pmatrix}$$

Notice that these two eigenvectors are complex conjugates since the eigenvalues are complex conjugates. So it is only necessary to compute one eigenvector and we get the other one for free!

18) The eigenvalues are,

$$\begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(4 - 4\lambda + \lambda^2 - 1) = (1-\lambda)(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 3.$$

Then the eigenvectors are,

$$\begin{pmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} x = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x_2,\ x_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Homework Tip. Lets suppose we have some factored polynomial: $(x+2)(x-1)(x-3)$, and we want to figure out for what intervals $(x+2)(x-1)(x-3) = 0$, $(x+2)(x-1)(x-3) < 0$, or $(x+2)(x-1)(x-3) > 0$. The zeroes are the easiest since they are just the roots of the solution, so we can plot $x = -2, 1, 3$ on a number line and test a point in each interval $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, $(3, \infty)$ to see if the polynomial is positive or negative.