10.1: Linear Systems of ODEs

We study differential equations because real world processes are governed by rates. You have all seen first order and even higher order ODEs representing one process, but what if there are multiple coupled processes? Then we need an array of ODEs,

$$\begin{aligned}
\dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\dot{x}_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

This can be written as the matrix equation,

$$\frac{d}{dt}\overrightarrow{x} = A(t)\overrightarrow{x} + \overrightarrow{f}(t) \tag{1}$$

Then the solution to the ODE involves two parts, a characteristic part that solves the homogeneous equation, and a particular part that supplements the solution to match the forcing on the right hand side. That is, $\overrightarrow{x} = \overrightarrow{x}_c + \overrightarrow{x}_p$. Here $\overrightarrow{x}_c = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)} + \cdots + c_n\overrightarrow{x}^{(n)}$; i.e., it is a superposition of n solutions since we have n ODEs, and this solves the homogeneous equation

$$\frac{d}{dt}\overrightarrow{x} = A(t)\overrightarrow{x} \tag{2}$$

Now lets go over some definitions that you may have seen before, but it's worth revising.

Definition 1. The set of functions $\{h_1, h_2, \ldots, h_{n-1}, h_n\}$ are said to be linearly independent if $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n \neq 0$, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n$ is said to be a linear combination of $h_1, h_2, \ldots, h_{n-1}, h_n$.

Theorem 1. *If $\overrightarrow{x}^{(1)}, \overrightarrow{x}^{(2)}, \ldots, \overrightarrow{x}^{(n-1)}, \overrightarrow{x}^{(n)}$ are solutions to (2), then any linear combination of $\overrightarrow{x}$'s are also solutions.*

For example, $\overrightarrow{x} = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)}$, $y = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)} + \cdots + c_n\overrightarrow{x}^{(n)}$, etc. are also solutions.

Now we define what the Wronskian is, which you have seen before, but in a slightly different context.

Definition 3. The *Wronskian* is

$$W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n)} \\ \ldots & \ldots & \vdots & \ldots \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} \end{vmatrix} \tag{3}$$

Theorem 2. *Suppose $\overrightarrow{x}^{(1)}, \overrightarrow{x}^{(2)}, \ldots, \overrightarrow{x}^{(n-1)}, \overrightarrow{x}^{(n)}$ are solutions to (2) on an interval I, with the usual initial conditions, then $W \neq 0$ guarantees they are linearly independent on $\in I$.*

So the rewording of the above theorem implies that if the Wronskian is zero at a single point then the function may still be linearly independent.

Definition 4. The set of all linearly independent solutions of an ODE is called the fundamental set of solutions for that ODE.

10.2: Homogeneous Linear Systems

Consider the 2×2 ODE $\frac{d\vec{x}}{dt} = A(t)\vec{x}$. As we have seen with the linear algebra problems, a 2-dimensional eigenvalue problem has three types of solutions:

Case	Eigenvalue	Eigenvector	General Solution
Real Distinct	$A\vec{v} = \lambda\vec{v} \Rightarrow \lambda = \lambda_1,\, \lambda_2$	$\vec{v} = \vec{v}_1,\ \vec{v}_2$	$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$
Real Repeated	$A\vec{v} = \lambda\vec{v} \Rightarrow \lambda = \lambda$	$\vec{v} = \vec{v},\ A\vec{w} = \vec{v}$	$\vec{x} = c_1\vec{v}e^{\lambda t} + c_2 e^{\lambda t}\left[\vec{v}t + \vec{w}\right]$
Complex Conjugate	$\lambda = \xi \pm i\theta$	$\vec{v} = \vec{\nu} \pm i\vec{\omega}$	$\vec{x} = c_1 e^{\xi t}\left[\vec{\nu}\sin\theta t + \omega\cos\theta t\right] + c_2 e^{\xi t}\left[\vec{\nu}\cos\theta t - \vec{\omega}\cos\theta t\right]$

Now, lets go over a bunch of examples

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\vec{x}$

Solution: First find the eigenvalues,

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 6 - \lambda + \lambda^2 + 4 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = -1, 2$$

Then we find the eigenvectors,

$$\begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}\vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}.$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\vec{x}$

Solution: The eigenvalues are

$$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = \lambda^2 - 9 + 9 = \lambda^2 = 0 \Rightarrow \lambda = 0.$$

The eigenvector is,

$$\begin{pmatrix} 3 & 1 \\ 9 & -3 \end{pmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

And the generalized eigenvector is

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\vec{w} = \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]$$

Ex: Consider the system of ODEs $\dot{x} = 2x - 5y$, $\dot{y} = x - 2y$.

Solution: This translates into the matrix ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x}$. We take the eigenvalues as usual,

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (4-\lambda^2) + 5 = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Then the eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}; \qquad \vec{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

Then our solution is

$$\hat{x} = c_1\begin{pmatrix} 2+i \\ 1 \end{pmatrix}e^{it} + c_2\begin{pmatrix} 2-i \\ 1 \end{pmatrix}e^{-it} = c_1\begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i\sin t) + c_2\begin{pmatrix} 2-i \\ 1 \end{pmatrix}(\cos t - i\sin t)$$

$$= c_1\left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}\right] + c_2\left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\cos t - 2\sin t \\ -\sin t \end{pmatrix}\right]$$

So, our real solution would be,

$$x = (c_1 + c_2)\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + (c_1 - c_2)\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Notice, since the eigenvectors are complex conjugates, we only need one eigenvector to find our solution. This is what we will do from now on. **Don't do the problem the way I showed the instructional example!**