10.2: Homogeneous Linear Systems (continued)

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \vec{x}$; $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution: Here we will solve an IVP. We take the eigenvalues,

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = -3 + 2\lambda + \lambda^2 + 5 = \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) = -1 \pm i.$$

The eigenvectors for $\lambda = -1 + i$ is,

$$x^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \Rightarrow \hat{x} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i \sin t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Then our general solution is,

$$x = Ae^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + Be^{-t} \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Now we plug in our initial conditions,

$$x(0) = A \begin{pmatrix} 2 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow A = 1 \Rightarrow B = -1$$

Then our solution is,

$$x = e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$; $\vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solution: Again we find the eigenvalues

$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0 \Rightarrow \lambda = 2,\, 4.$$

Then we find the eigenvectors

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Our general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

Then we solve for the constants

$$\vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow c_1 = -3/2 \Rightarrow c_2 = 7/2$$

Then the full solution is

$$\vec{x} = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

Ex: $\frac{d\vec{x}}{dt} = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \vec{x}$

Solution: The eigenvalues are

$$\begin{vmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 36 = (\lambda-6)^2 = 0 \Rightarrow \lambda = 6.$$

Then the eigenvector and the generalized eigenvector are

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \qquad \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{w} = \vec{v} \Rightarrow \vec{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Then the general solution is

$$\vec{x} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 e^{6t} \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Additional Examples. Here are some additional examples for **distinct real eigenvalues** that we didn't do in class. You must understand how to do these problems before you can hope to solve the the distinct real and complex conjugate cases, so if you are having trouble please go through these examples step by step.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and comment on what happens as $t \to \infty$ for $c_2 = 0$ and $c_2 \neq 0$.

Solutions: Again we find the eigenvalues,

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

The eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Here if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$ and comment on what happens as $t \to \infty$.

Solution: Again,

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = (\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = -1, \, -3.$$

And the eigenvectors are,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Our solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

Here $x \to 0$.

Ex: Solve the ODE $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \vec{x}$ and comment on what happens as $t \to \infty$ for $c_2 = 0$ and $c_2 \neq 0$.

Solutions: Again the eigenvalues are

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 6 = -6 - \lambda + \lambda^2 + 6 = \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0, \, 1$$

with the eigenvectors,

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Then our solution is

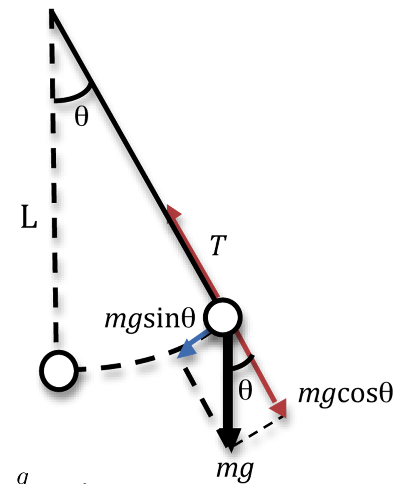
$$x = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

So our solution behaves as follows: if $c_2 = 0$, $x = c_1(-2, 1)$; i.e. the first eigenvector. If $c_2 \neq 0$, $x \to \infty$.

Sometimes we just can't solve problems exactly. In fact most problems don't have exact solutions. These equations are either solved numerically, in which case the numerics might get it wrong, or we extract information form the equations without solving.

Consider the pendulum. We will consider both the frictional and frictionless case. **For the exam just to keep things simple we will just focus on the frictionless case**, but it is important to understand what happens in the frictional case as well.

In order to derive the ODE we need the force along its arc. Then we use Newton's law: $F = ma$. To find the acceleration a along the arc lets consider the arc length: $s = L\theta$, then the velocity is $v = L\frac{d\theta}{dt} = \dot{\theta}$, and the acceleration is $a = L\frac{d^2\theta}{dt^2} = \ddot{\theta}$, which gives us a force of $F = m\ddot{\theta}$. Now we need to figure out what F is. This consists of the force from gravitational acceleration and damping from friction, $F = -\nu L\dot{\theta} - mg\sin\theta$. This gives us the ODE



$$mL\frac{d^2\theta}{dt^2} = -\nu L\frac{d\theta}{dt} - mg\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{\nu}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

But this is really ugly, so lets simplify the equation a bit,

$$\ddot{\theta} = -\gamma\dot{\theta} - k\sin\theta \tag{1}$$

Higher order equations are tough to deal with, especially when they are nonlinear, so lets change this into a system of first order equations by letting $\omega = \dot{\theta}$

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\gamma\omega - k\sin\theta$$