12.1: Orthogonality

For these spaces we can find many analogues to vector spaces. The difference is that now we are thinking in terms of infinite dimensions. In order to ease the transition I will provide analogues for most definitions.

Definition 1. We call the functional

$$< u, v >= \int_\alpha^\beta u(x)v(x)dx, \tag{1}$$

the L^2 inner product on the interval $[\alpha, \beta]$.

This is analogous to the *dot product*.

Definition 2. If $< u, v >= 0$ we say u and v are orthogonal.

This is like perpendicular vectors. Recall, two vectors a and b are perpendicular if $a \cdot b = 0$.

Definition 3. We call $||u|| =< u, u >$ the L^2 norm of u.

Definition 4. If $||u|| = 1$ we say u is normal.

Definition 5. If $||u|| = 1$ and $||v|| = 1$ and $< u, v >= 0$ we say that u and v are orthonormal.

In class we did the following examples,

4) We take the inner product via the u-sub $u = \sin x \Rightarrow du = \cos x dx$,

$$\langle f_1, f_2\rangle = \int_0^\pi \cos x \sin^2 x dx = \int_a^b u^2 du = \frac{1}{3}u^3\bigg|_a^b = \frac{1}{3}\sin^3 x\bigg|_0^\pi = 0.$$

So the two functions are orthogonal.

10) **Orthogonality:** This one is a bit harder because we have general arguments in the sines, but the form of the arguments will be enough for us to take the inner product.

$$\left\langle \sin\frac{n\pi x}{p}, \sin\frac{m\pi x}{p}\right\rangle = \int_0^p \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}dx.$$

Now we need to recall some trig identities:

Trig Identities.

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \end{aligned} \tag{2}$$

So

$$\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \Rightarrow \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p} = \frac{1}{2}\left[\cos\left((n-m)\frac{\pi}{p}x\right) - \cos\left((n+m)\frac{\pi}{p}x\right)\right]$$

We can do the integrals individually,

$$\int_0^p \cos\left((n-m)\frac{\pi}{p}x\right)dx = \frac{p}{\pi(n-m)}\sin\left((n-m)\frac{\pi}{p}x\right)\bigg|_0^p = 0$$

$$\int_0^p \cos\left((n+m)\frac{\pi}{p}x\right)dx = \frac{p}{\pi(n+m)}\sin\left((n+m)\frac{\pi}{p}x\right)\bigg|_0^p = 0$$

$$\Rightarrow \left\langle \sin\frac{n\pi x}{p}, \sin\frac{m\pi x}{p}\right\rangle = 0 \text{ for } n \neq m,$$

so the functions constitute an orthogonal set on $[0, p]$.

Normality: No we find the norm by letting $n = m$,

$$\left|\left|\sin\frac{n\pi x}{p}\right|\right|^2 = \left\langle \sin\frac{n\pi x}{p}, \sin\frac{n\pi x}{p}\right\rangle = \int_0^p \sin^2\frac{n\pi x}{p}dx \tag{3}$$

For this we need the double angle identities

Trig Identities.

$$\begin{aligned}\sin^2\theta &= \frac{1}{2}\left(1-\cos 2\theta\right)\\ \cos^2\theta &= \frac{1}{2}\left(1+\cos 2\theta\right)\end{aligned} \tag{4}$$

Then our integral becomes

$$\frac{1}{2}\int_0^p \left(1-\cos\frac{2n\pi x}{p}\right)dx = \frac{1}{2}\left[x-\frac{p}{2n\pi}\sin\frac{2n\pi x}{p}\right]_0^p = \frac{1}{2}p \Rightarrow \left\|\sin\frac{n\pi x}{p}\right\| = \sqrt{\frac{p}{2}}$$