12.2: FOURIER SERIES

This definition allows us to construct a space of functions out of two simple functions. Now equipped with our new machinery we can derive a series representation that is ideal for periodic functions. We did this in class, but here I shall just remind you of the formulas:

Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right];$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right),\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

Now lets do some problems. While a lot of these want plotting, we did them in class, so I won't show them here, but make sure you know how to plot these things.

Ex: Find the Fourier Series of the function

$$f(x) = \begin{cases} 1 & -L < x < 0, \\ 0 & 0 \le x < L; \end{cases}$$

(a) Sketch it!
(b) We first do a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{0} dx = 1.$$

Notice that we always do a_0 separately. Then we do a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} \to 0$$

Finally, for b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0}$$
$$= -\frac{1}{n\pi} + \frac{1}{n\pi}\cos(n\pi) = \frac{-1+(-1)^n}{n\pi} = -\frac{2}{n\pi}\begin{cases} 1 & n \text{odd, i.e. } n = 2k+1;\ k = 0, \pm 1, \pm 2, \dots \\ 0 & n \text{even, i.e. } n = 2k;\ k = 0, \pm 1, \pm 2, \dots \end{cases}$$

Then our Fourier series becomes

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{1}{L}(2k+1)\pi x\right).$$

Ex: Find the Fourier Series of the function $f(x) = x^2/2$ on $[-2, 2]$
(a) Plot it!
(b) Again, we do a_0 first

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}dx = \frac{x^3}{12}\Big|_{-2}^{2} = \frac{4}{3}.$$

Now to do a_n we need to do by parts twice, which you can do yourselves. I'll just give the final form of the antiderivative.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 \frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{(n\pi)^3}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 = \frac{8}{(n\pi)^2}\cos(n\pi) = (-1)^n\frac{8}{(n\pi)^2}.$$

For b_n we get

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\sin\left(\frac{n\pi x}{L}\right)dx = 0.$$

because we are integrating an odd function on a symmetric interval. Then our Fourier series is

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

15) This is a book problem.

First we find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}e^x\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^p i - e^{-\pi}\right) = \frac{2}{\pi}\sinh\pi$$

Then we find a_n via "by parts" using $u = \cos nx \Rightarrow du = -n\sin nx dx$ and $dv = e^x dx \Rightarrow v = e^x$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx = \frac{1}{\pi}\left[e^x\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x\sin x dx\right]$$

Then we do another by parts: $u = \sin nx \Rightarrow du = n\cos nx$ and $dv = e^x dx \Rightarrow v = e^x$

$$\frac{1}{\pi}\left\{e^x\cos nx\Big|_{-\pi}^{\pi} + n\left[\cancelto{0}{e^x\sin nx\Big|_{-\pi}^{\pi}} - n\int_{-\pi}^{\pi} e^x\cos nx dx\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left(e^{\pi} - e^{-\pi}\right)(-1)^n - n^2\int_{-\pi}^{\pi} e^x\cos nx dx\right\} = (-1)^n\frac{2}{\pi}\sinh\pi - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx$$

Now we notice that we have $\int_{-\pi}^{\pi} e^x\cos nx dx$ on both the right and left hand sides, so we can combine them,

$$\frac{n^2+1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx = (-1)^n\frac{2}{\pi}\sinh\pi \Rightarrow a_n = \frac{(-1)^n}{n^2+1}\cdot\frac{2}{\pi}\sinh\pi$$

For b_n we have something similar so I will skip a bunch of steps,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx dx = \frac{1}{\pi}\left\{-e^x\sin nx\Big|_{-\pi}^{\pi} - n\left[e^x\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x\sin nx dx\right]\right\}$$

$$\Rightarrow \frac{n^2+1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx dx = -(-1)^n\frac{2n}{\pi}\sinh\pi \Rightarrow b_n = -\frac{(-1)^n}{n^2+1}\cdot\frac{2n}{\pi}\sinh\pi$$

Then the Fourier Series is

$$f(x) = \frac{2}{\pi}\sinh\pi\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right].$$