12.3: Even and Odd Functions

As we saw for the last problem in the preceding section, it can be useful to know whether or not a function is odd or even. Also, many times we will want the Fourier series of a non-periodic function. In order to do this we need to create a periodic function that includes our non-periodic function. Instead of creating something that is neither odd nor even if we create an even or odd function we can save a lot of time. Before we see these techniques lets define some terms and develop the theory.

Definition 1. Consider the function $f(x)$ such that $f(-x) = f(x)$, then f is said to be even.

Definition 2. Consider a function $f(x)$ such that $f(-x) = -f(x)$, then f is said to be odd.

There are some important properties that we should keep in mind.

Properties.

- Sum/difference of two even functions is even.
- Sum/difference of two odd functions is odd.
- Sum/difference of an even and an odd function is neither even nor odd.
- Product/quotient of two even functions is even.
- Product/quotient of two odd functions is even.
- Product/quotient of an even function and an odd function is odd.
- If f is even, $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.
- If f is odd, $\int_{-L}^{L} f(x)dx = 0$.

Now we can think of a Fourier cosine series and Fourier sine series. These can be derived straight from the Fourier series equations so it's best not to memorize these formulas.

Fourier cosine series. If f is an even periodic function generated on $-L \le x \le L$, then $b_n = 0$, so

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{1}$$

Fourier sine series. If f is an odd periodic function generated on $-L \le x \le L$, then $a_n = 0$, so

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{2}$$

For the next few problems we just apply the definition of odd and even functions.

(1) Odd 5) Even 6) Neither

Periodic Extensions. Suppose a function f is defined only on $[0, L]$. If we want to find the Fourier series of this we need to make a periodic function that "includes" f. These are called periodic extensions and can either be odd or even.

For these problems we did the sketching in class. Here I will do the problems that requires calculations

Ex: Find the Fourier Sine Series of $f(x) = L - x$ on $[0, L]$.

(a) Notice that for odd extensions our periodic function of period $2L$ becomes

$$g(x) = \begin{cases} -f(-x) & -L < x < 0, \\ f(x) & 0 < x < L; \end{cases} = \begin{cases} -L - x & -L < x < 0, \\ L - x & 0 < x < L; \end{cases}$$

We know that for odd extensions we'll get a sine series so we only do the sine calculations,

$$b_n = \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)dx = -(L-x)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{n\pi} + \cancelto{0}{\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L}$$

Then our Fourier sine series is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

(b) Sketch the solution for $L = 4$.

Ex: Find the Fourier Sine and Cosine series of the following function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 0 & \text{for } 1 < x < 2 \end{cases}$$

(a) Sketch the even and odd extensions of the function.

(b) For the cosine series we have

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \int_0^1 x dx = \frac{1}{2}.$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1$$
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}.$$

Notice that for this problem we can't simplify the indices in any reasonable manner, so we leave it as is. So the Fourier cosine series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\right)\cos\left(\frac{n\pi x}{2}\right).$$

Now, for the sine series we have

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$
$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

Then our Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)$$