12.5 Boundary Value Problems

We are used to initial value problems where we are given initial data. What if we are given boundary data instead? There are many applications where things are happening for a long period of time and we don't know what happened in the beginning, but we do know something about the boundary. The usual problems are solved in a similar fashion to Initial Value Problems. We do however have a bit more theory.

Definition 1. The boundary values (for a second order ODE) $y(a)$, $y(b)$, $y'(a)$, and/or $y'(b)$ are said to be homogeneous if any two of the above boundary data are zero.

We also have eigenvalue problems for BVPs. Recall that for matrices the eigenvalue problems were of the form $Ax = \lambda x$, where we solve for the "eigenvalue", λ. For BVPs of a second order ODE, we consider our linear operator to be $L = d^2/dx^2$ (for matrices the linear operator is the matrix A). So we wish to solve the problem $Ly = \lambda y$; i.e. $y'' + \lambda y = 0$. Here the $y_n's$ corresponding to $\lambda_n's$ are called eigenfunctions (similar to eigenvectors in the matrix case). We notice that eigenvalue problems are only for homogeneous boundary data.

Definition 2. The boundary value problem

$$y'' + \lambda y = 0; \qquad \text{(with homogeneous boundary conditions)}, \tag{1}$$

is called an eigenvalue problem. And the nontrivial (i.e. $y_n \neq 0$) solutions y_n corresponding to λ_n are the eigenfunctions of the corresponding eigenvalues.

Now lets do some boundary value problems,

Ex: $y'' + y = 0$; $y'(0) = 1$, $y(L) = 0$.

Solution: The characteristic polynomial gives us

$$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y = A\cos t + B\sin t \Rightarrow y' = -A\sin t + B\cos t.$$

Then our first boundary condition gives $y'(0) = B = 1$, and

$$y(L) = A\cos L + \sin L = 0 \Rightarrow A = -\tan L;\ L \neq (2k+1)\frac{\pi}{2},\ k = 0,\ \pm1,\ \pm2,\ \ldots$$

However, if $\cos L = 0$, $\sin L = 0$, but this is clearly false because $\sin x \neq 0$ when $\cos x = 0$ and vice-versa, so the BVP has no solution if $L = (2k+1)\frac{\pi}{2}$.

Ex: $y'' + \lambda y = 0$; $y'(0) = y'(\pi) = 0$.

Solution:

(i) If $\lambda > 0$, let $\lambda = \mu^2$. Then

$$r = \pm i\mu \Rightarrow y = A\cos\mu t + B\sin\mu t \Rightarrow y' = -A\mu\sin\mu t + B\mu\cos\mu t$$

From the first boundary condition we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ because $\lambda > 0$. From the second B.C. we get $y'(\pi) = -A\mu\sin\mu\pi = 0$. Since we don't want trivial solutions if we can avoid them we can't have $A = 0$, so we require $\sin\mu\pi = 0$ then $\mu = n\pi$ where $n = 1, 2, \ldots$, so our eigenfunctions for the corresponding eigenvalues are

$$y_n = \cos n\pi t;\ \lambda_n = n^2,\ n = 1, 2, \ldots$$

(ii) If $\lambda < 0$, let $\lambda = -\mu^2$. Then

$$r = \pm\mu \Rightarrow y = c_1 e^{\mu t} + c_2 e^{-\mu t} = A\cosh\mu t + B\sinh\mu t \Rightarrow y' = A\sinh\mu t + B\cosh\mu t.$$

The B.C.'s give $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y'(\pi) = A\sinh\mu\pi = 0$, but sinh is only zero at zero and $\mu \neq 0$ since $\lambda < 0$, so we have $A = 0$. Then $y \equiv 0$, so unfortunately we get a trivial solution.
(iii) If $\lambda = 0$, $y = c_1 x + c_0 \Rightarrow y' = c_1$, then applying the B.C.'s give $y'(0) = c_1 = 0$ and $y'(\pi) = 0$ automatically. Then our eigenvalue and eigenfunction are

$$y_0 = 1,\ \lambda_0 = 0.$$

Notice I left out the constants. It is up to you if you want to include it or not.

Complete Solution:

$$y = c_0 + \sum_{n=1}^{\infty} A_n \cos n\pi t$$

Ex: $y'' + \lambda y = 0$; $y'(0) = y(L) = 0$

Solution:

(i) If $\lambda > 0$, let $\lambda = \mu^2$, then

$$y = A\cos\mu t + B\sin\mu t \Rightarrow y' = -A\mu\sin\mu t + B\mu\cos\mu t.$$

Notice how we have the same exact general solution! You do enough of these problems and you can go straight to the solution and it's derivative without having to do the characteristic polynomial. Now, from the B.C.'s we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y(L) = A\cos\mu L = 0$. So we require $\mu = (2n-1)\pi/2L$ where $n = 1, 2, 3, \ldots$, then our eigenvalues and eigenfunctions are

$$y_n = A_n \cos\left((2n-1)\frac{\pi}{2}t\right);\ \lambda_n = (2n-1)^2\frac{\pi^2}{4},\ n = 1, 2, 3, \ldots$$

(ii) If $\lambda < 0$, let $\lambda = -\mu^2$, then

$$y = A\cosh\mu t + B\sinh\mu t \Rightarrow y' = A\mu\sinh\mu t + B\mu\cosh\mu t.$$

From the B.C.'s we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y(L) = A\cosh\mu L = 0 \Rightarrow A = 0$, again it's the trivial solution $y \equiv 0$.
(iii) If $\lambda = 0$, $y = c_1 x + c_0 \Rightarrow y' = c_1$, from the B.C.'s we get $y'(0) = c_1 = 0$ and $y(L) = c_0 = 0$, so again we have the trivial solution $y \equiv 0$.

Complete Solution:

$$y = \sum_{n=1}^{\infty} A_n \cos\left((2n-1)\frac{\pi}{2}t\right)$$

The Heat Equation. Consider heat conduction in some bulk space V with a boundary ∂V. Also consider an infinitesimal space in that bulk called dV. Let $u(x, y, z, t)$ represent the temperature in V at any time t. Let $E = c\rho u$ where c is the specific heat and ρ is the mass density of the bulk, be the total energy in dV.

There are some fundamental laws that will lead us to the heat equation:

> Fourier heat conduction laws:
> (1) If the temperature in a region is constant, there is no heat transfer in that region.
> (2) Heat always flows from hot to cold.
> (3) The greater the difference between temperatures at two points the faster the flow of heat from one point to the other.
> (4) The flow of heat is material dependent.

All these laws can be summarized into one equation

$$\phi(x, y, z, t) = -K_0 \nabla u(x, y, z, t) \tag{2}$$

Now we can form a word equation:

$$\text{(Rate of change of heat)} = \text{(Heat flowing into } dV \text{ per unit time)} + \text{(Heat generated in } dV \text{ per unit time)} \tag{3}$$

The first statement is the rate of change of the total energy E. The second is the flux at ∂V in the normal direction. The third is additional heat being generated in dV. For the third statement lets called the additional heat Q. This gives us the equation

$$\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = -\oiint_{\partial V} \phi \cdot n \, dS + \iiint_V Q \, dV \tag{4}$$

And using divergence theorem we get

$$\oiint_{\partial V} \phi \cdot n \, dS = \iiint_V \nabla \cdot \phi \, dV = \iiint_V \nabla \cdot (-K_0 \nabla u) \, dV = K_0 \iiint_V \nabla^2 u \, dV$$

therefore, the equation becomes

$$\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = \iiint_V c\rho \frac{\partial}{\partial t} u \, dV = K_0 \iiint_V \nabla^2 u \, dV + \iiint_V Q \, dV \Rightarrow c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q. \tag{5}$$

If we consider the case $Q = 0$; i.e., no external heat being generated, and if we divide through by $c\rho$, then we get the simplest form of the heat equation

$$\frac{\partial u}{\partial t} = K \nabla^2 u \tag{6}$$

where K is called the thermal diffusivity. In 1-D this is,

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \tag{7}$$

The Wave Equation. Here we will only derive the 1-D version, but keeping in mind we can extend the notion of a derivative to higher dimensions with the ∇ operator.

Consider a vibrating string. Let u be the vertical displacement. The slop of the string at any horizontal position x is $\tan\theta(x) = \partial u/\partial x$ where θ is the angle from the horizontal.

If we ignore any horizontal motion of the atoms we can directly use Newton's laws for the vertical motion. The mass of any small segment of string is $\rho\Delta x$ and the acceleration is $\partial^2 u/\partial t^2$. The total tensile force on the string at x is $F(x,t)$ and at Δx is $F(x+\Delta x, t)$. We can "pick out" the vertical component of the force by multiplying it by sine of the angle. Lets also consider an external force on the string Q. Then we get the equation

$$\rho\Delta x \frac{\partial^2 u}{\partial t^2} = F(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - F(x,t)\sin(\theta(x,t)) + \rho\Delta x Q(x) \tag{8}$$

Dividing through by Δx and taking the limit gives us

$$\rho\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(F(x+\Delta x,t)\sin(\theta(x+\Delta x,t)) - F(x,t)\sin(\theta(x,t))\right)+\rho Q(x) = \frac{\partial}{\partial x}\left(F(x,t)\sin(\theta(x,t))\right)+\rho Q(x)$$

If our angle of deflection isn't huge, $\tan\theta = \sin\theta/\cos\theta \approx \sin\theta$, so we can approximate sine by $\sin\theta = \partial u/\partial x$. Let us also assume a constant tensile force through out the string; i.e., $F = T_0$. Then our equation simplifies to

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T_0\frac{\partial u}{\partial x}\right) + \rho Q \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho}\frac{\partial^2 u}{\partial x^2} + Q \tag{9}$$

Here $T_0/\rho = c^2$, where c is the speed of propagation of the wave. Now, if the only external force is gravity, it is negligible compared to T_0, so $Q \approx 0$. This gives us the simplest wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{10}$$

This can be extended to the higher orders using the ∇ operator

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \tag{11}$$

Laplace's Equation. We will often want to see what happens for steady state problems or when the time derivatives of a PDE are zero. This gives us Laplace's Equation

$$\nabla^2 u = 0 \tag{12}$$

Boundary and Initial Condition. Consider a boundary $x = b$, then we have some usual conditions

Dirichlet conditions: $u(b,t) = B$ [Heat: prescribed, Wave: clamped]

Neumann conditions: $u_x(b,t) = B$ [Heat: flux, Wave: sloped string]

We may also have a combination of these conditions. It should be noted that for the heat equation because we have one time derivative, it will have one initial condition. For the wave equation since we take two time derivatives we will have two initial conditions: one for the initial profile and the other for the initial velocity.

Examples.

6) Here we need Newton's law of cooling: The rate of change of temperature at a point is proportional to the difference between that and the surrounding temperature; i.e., $\partial T/\partial t = -K(T - T_a)$, where T_a is the ambient temperature. Since the heat transfer is not happening in the domain, but rather lateral to the domain we treat this as external heat generation. The problem also has insulated boundaries ($u_x = 0$ at the boundaries), and the initial temperature is a constant 100 C. So we get

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - K(u - 50); \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0; u(x,0) = 100.$$

10) They say we have an external force proportional to the position, so $Q = kx$, and the problem tells us that the string is secured at the ends ($u = 0$ at the boundaries). For the initial condition it tells us that it is at rest ($u_t(x,0) = 0$) on the x-axis ($u(x,0) = 0$). So we have

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + kx; u(0,t) = u(L,t) = 0; u(x,0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

12) Here the one complication is the boundary on the right. We need a way to express that it is 100C only after a certain value of y. We do this using the Heaviside function

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; u(0,y) = e^{-y}, u(x,0) = f(x), u(\pi,y) = 100\left(1 - H_1(y)\right).$$