Week 10 Part 2: Heat Equation Example and Finite Differences for Heat Equation

Heat Equation Examples. Consider the heat equation with a generic initial condition,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = f(x). \tag{1}$$

with the following boundary conditions: $u(0,t) = u(L,t) = 0$.

Solution: We make the Ansatz, $u(x,t) = T(t)X(x)$. Then we plug this into our heat equation

$$u_t = T'(t)X(x),\ u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}.$$

Since the LHS is a function of t alone, and the RHS is a function of x alone, and since they are equal, they must equal a constant. Lets call it $-\lambda^2$. Then we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \tag{2}$$

Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue λ^2. Now we must solve the two differential equations.

The T equation is the easiest to solve

$$\frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \frac{dT}{dt} = -k\lambda^2 T \Rightarrow \frac{dT}{T} = -k\lambda^2 dt \Rightarrow \int \frac{dT}{T} = \int -k\lambda^2 dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t}$$

Notice that we don't include the constant in front of the exponential, and that is because the X equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don't have to though.

Now, we solve the X equation by recalling our Sturm-Liouville problems

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0.$$

If we look at the $\lambda = 0$ case we have $X(0) = c_2 = 0$ and $X(L) = Lc_1 = 0$, so $X \equiv 0$.

Now we look at the $\lambda \neq 0$ case. $X(0) = A = 0$ and

$$X(L) = X(L) = B\sin\lambda x = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin\frac{n\pi}{L}x \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Next we combine the T and X solutions to get the general solutions,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{3}$$

And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Then our full solution is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{4}$$

Finite Differences. The idea here is that we know how to solve boundary value problems using finite differences so we are just going to solve a boundary value problem at each timestep n and iterate using a loop.

We must discretize the Heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2};$$

using finite differences with i representing the spatial steps and n representing the temporal steps.

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

which is just the usual forward difference. And

$$k\frac{\partial^2 u}{\partial x^2} = k\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2}\right]$$

which is the average of the second derivative central difference for the next time and the previous time. Plugging this into the PDE gives us

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2}\right] \tag{5}$$

which can be simplified by letting $\mu = k\Delta t/(2\Delta x^2)$,

$$u_i^{n+1} - u_i^n = \mu\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right]$$

then we put all the time $n+1$ terms on the left hand side and all the time n terms on the right hand side

$$-\mu u_{i+1}^{n+1} + (1+2\mu)u_i^{n+1} - \mu u_{i-1}^{n+1} = \mu u_{i+1}^n + (1-2\mu)u_i^n + \mu u_{i-1}^n$$

Just like we did with ODEs, we can use tridiagonal matrices to represent this. First lets write out each equation so we get a good idea of what needs to go into the tridiagonal entries. It should be noted that u_0 and u_M are just boundary terms, so lets call them u_a and u_b (the left and right hand boundary values).

$$\begin{aligned}
(1+2\mu)u_1^{n+1} - \mu u_2^{n+1} &= \mu u_2^n + (1-2\mu)u_1^n + 2\mu u_a \\
-\mu u_1^{n+1} + (1+2\mu)u_2^{n+1} - \mu u_3^{n+1} &= \mu u_1^n + (1-2\mu)u_2^n + \mu u_3^n \\
&\vdots \\
-\mu u_{i-1}^{n+1} + (1+2\mu)u_i^{n+1} - \mu u_{i+1}^{n+1} &= \mu u_{i-1}^n + (1-2\mu)u_i^n + \mu u_{i+1}^n \\
&\vdots \\
-\mu u_{M-3}^{n+1} + (1+2\mu)u_{M-2}^{n+1} - \mu u_{M-1}^{n+1} &= \mu u_{M-3}^n + (1-2\mu)u_{M-2}^n + \mu u_{M-1}^n \\
-\mu u_{M-2}^{n+1} + (1+2\mu)u_{M-1}^{n+1} &= \mu u_{M-2}^n + (1-2\mu)u_{M-1}^n + 2\mu u_b
\end{aligned}$$

Then all we have is an $Au^{n+1} = b$ equation where $b = Bu^n + (2\mu u_a, 0, 0, \dots, 0, 0, 2\mu u_b)$, and

$$A = \begin{pmatrix} 1+2\mu & -\mu & & & \\ -\mu & 1+2\mu & -\mu & & \\ & \ddots & \ddots & \ddots & \\ & & -\mu & 1+2\mu & -\mu \\ & & & -\mu & 1+2\mu \end{pmatrix}; \quad B = \begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & 1-2\mu & \mu \\ & & & \mu & 1-2\mu \end{pmatrix} \tag{6}$$

and therefore

$$b = Bu^n + \begin{pmatrix} 2\mu u_a \\ 0 \\ \vdots \\ 0 \\ 2\mu u_b \end{pmatrix} \tag{7}$$