Week 4 Part 3: Iterative Schemes

We talk more about iterative methods in the coding lectures, but I derive the theory here.

Scalar iteration

Suppose we want to solve $(1-m)x = b; |m| < 1$, then clearly $x = b/(1-m)$. However, we want to see how these numerical methods are derived, so lets pretend that we do not know the solution. Notice that $(1-m)x = b$, then

$$x = mx + b \tag{1}$$

This is just the equation of a fixed oint of a recurrence relation. If the recurrence is stable,

$$x_{n+1} = mx_n + b \tag{2}$$

will approximate the solution. If we assume that $x_0 = 0$ we get

$$x_1 = b \Rightarrow x_2 = mb + b \Rightarrow \cdots \Rightarrow x_n = m^{n-1}b + m^{n-2}b + \cdots + m^2b + mb + b.$$

This is just a geometric series in m, so

$$x_n = \frac{1-m^n}{1-m}b. \tag{3}$$

Notice that if $|m| < 1$,

$$x_n \to \frac{b}{1-m} \qquad \text{as} \qquad n \to \infty$$

Matrix iteration

Similarly, for matrices, we can do $x_{n+1} = Mx_n + b$, but this gives us the fixed point equation

$$x = Mx + b \Rightarrow (I - M)x = b. \tag{4}$$

If $A = I - M$, then this is our infamous $Ax = b$ problem. Notice that M^n has to be decreasing as $n \to \infty$ for our algorithm to converge. Also, $I - M$ must be nonsingular.

Jacobi Method

Let $A = L + D + U$, then

$$Ax = (L + D + U)x = b \Rightarrow D^{-1}(L + D + U)x = D^{-1}b \Rightarrow x + D^{-1}(L + U)x = D^{-1}b,$$

which yields

$$x_{n+1} = -D^{-1}(L + U)x_n + D^{-1}b. \tag{5}$$

Here $M = -D^{-1}(L + U)$ and b is replaced with $D^{-1}b$.

Gauss-Seidel Method

Suppose L is faster to invert than A, and not much slower to invert than D, then we can invert $L + D$,

$$(L + D)^{-1}(L + D + U)x = (L + D)^{-1}b \Rightarrow x + (L + D)^{-1}Ux = (L + D)^{-1}b,$$

and

$$x_{n+1} = -(L + D)^{-1}Ux_n + (L + D)^{-1}b. \tag{6}$$

Convergence

Definition 1. An $n \times n$ matrix A is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1,\, i\neq j}^{n} |a_{ij}|, \quad \forall \quad 1 \leq i \leq n. \tag{7}$$

While we won't prove it in this class, but it can be shown that the Jacobi and Gauss-Seidel methods are guaranteed to converge only if the matrix is strictly diagonally dominant.