

SEC. 1.2 MATRICES AND VECTORS

Coefficient matrix

Consider

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \tag{1}$$

Then the matrix

$$A_{3 \times 3} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

is called the coefficient matrix of (1). Also, the matrix A is said to be a 3×3 matrix as it has 3 rows and 3 columns.Addition

Matrix addition works just like scalar addition, and we just add the respective elements together.

Consider

$$B_{3 \times 2} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}, C_{3 \times 2} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$B + C = \begin{bmatrix} 3 & 3 \\ 0 & 1 \\ 1 & 6 \end{bmatrix}$$

Further, the matrices B and C are of size 3×2 since they have 3 rows and 2 columnsScalar multiplication

Just multiply the scalar term with each element,

Ex:

$$2B = \begin{bmatrix} 4 & 2 \\ 6 & 0 \\ 0 & 8 \end{bmatrix}$$

Ex:

$$2C = \begin{bmatrix} 2 & 4 \\ -6 & 2 \\ 2 & 4 \end{bmatrix}$$

Ex:

$$2(B + C) = 2(C + B) = \begin{bmatrix} 6 & 6 \\ 0 & 2 \\ 2 & 12 \end{bmatrix}$$

Transpose

We denote transpose with a T superscripted onto the matrix,

Ex:

$$C^T = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Ex:

$$B^T = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Dot (inner) product

Most of you have seen dot products in your Calculus courses. We will see why this is called an inner product as well in later sections.

Ex:

$$\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 1 \cdot 2 + (-3) \cdot 3 + 1 \cdot 0 = -7.$$

Size

The size of a matrix is written as the number of rows by the number of columns, and if shown is given as a subscript.

Multiplication

We can only multiply matrices if the number of rows of the first is equivalent to the number of columns of the second. In order to multiply we will do the dot product of the row of the first matrix with the column of the second. Here is a generalization of that process for a 2×2 and 3×3 , and it is a simple extension to other sizes.

2×2 :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \quad (2)$$

3×3 :

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix} \quad (3)$$

Ex:

$$BC^T = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 4 & -5 & 4 \\ 3 & -9 & 3 \\ 8 & 4 & 8 \end{bmatrix}$$

Ex:

$$CB^T = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Notice that $BC^T = (CB^T)^T$ and vice-versa.

Ex:

$$C^T B = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{bmatrix} -7 & 5 \\ 7 & 10 \end{bmatrix}$$

Ex:

$$B^T C = \begin{bmatrix} -7 & 7 \\ 5 & 10 \end{bmatrix}$$

Notice that now that we know how to transpose products, we did not have to do any work since $B^T C = (C^T B)^T$.

Matrix form of linear systems

Now that we know how to multiply matrices we may convert our system of equations into a matrix equation, which the computer can understand. Lets again consider (1),

$$(1) \Rightarrow Ax = b \Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \quad (4)$$

We talked about a lot of notation, but we skipped some intentionally. We'll get back to those later.

We saw that matrices of different sizes cannot be commuted; i.e., $A_{2 \times 3} B_{3 \times 2} \neq B_{3 \times 2} A_{2 \times 3}$. However, with scalar multiplication we can commute (e.g., $5 \cdot 2 = 2 \cdot 5$), so can two matrices of the same size commute; i.e., $A_{2 \times 2} B_{2 \times 2} \stackrel{?}{=} B_{2 \times 2} A_{2 \times 2}$.

Ex: Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Notice

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix addition works just like scalar addition.

Matrix multiplication: $(AB)C = A(BC)$, $A(B + C) = AB + AC$, $AB \neq BA$. So, in general matrices do not commute. Is there a matrix that commutes with everything?

Consider the 2×2 matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

and a generic 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then $AI = A = IA$. This is multiplicative identity of matrices, and is called the identity matrix. For a general $n \times n$ matrix it takes the form,

$$I_n = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \tag{6}$$

that is, ones down the diagonal and zeros everywhere else, so for a 3×3 matrix it would be

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now lets look at properties of the transpose: $(A^T)^T = A$, $(A + B)^T = A^T + B^T$, $(cA)^T = c(A^T)$, $(AB)^T = B^T A^T$. Notice that a dot product of two vectors can also be written as $v \cdot w = v^T w$.