Sec. 1.3 Gaussian Elimination

Now lets go back and do a bunch of Gaussian Elimination problems. Again consider our equation from last time

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \tag{1}$$

we will write this as an augmented matrix by appending the right hand side (RHS) to the coefficient matrix,

$$\begin{matrix} \\ 2 \\ \\ \end{matrix}\begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 2 & | & 9 \end{bmatrix} = \begin{matrix} \\ \\ -1 \end{matrix}\begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ -2 & 7 & 2 & | & 9 \end{bmatrix} = \begin{matrix} \\ \\ -1 \end{matrix}\begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 8 & 3 & | & 14 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

This means $\boxed{w = 2}$, then we plug into the second equation to get $\boxed{v = 1}$, and finally the first to get $\boxed{u = 1}$.

The elements down the diagonal are called <u>pivots</u>. The augmented matrix is said to be in <u>row-echelon</u> form. The original matrix,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

is said to be in <u>upper triangular form</u>.

Here are a few more Gaussian elimination examples.

Ex:

$$\begin{matrix} \\ 3 \end{matrix}\begin{bmatrix} 1 & 3 & | & 11 \\ 3 & 1 & | & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & | & 11 \\ 0 & -8 & | & -24 \end{bmatrix} \Rightarrow \boxed{y = 3} \Rightarrow \boxed{x = 2};$$

Ex:

$$\begin{matrix} \\ -2 \end{matrix}\begin{bmatrix} -1 & 2 & | & 3/2 \\ 2 & -4 & | & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & | & 3/2 \\ 0 & 0 & | & -6 \end{bmatrix}$$

Clearly this matrix is singular, and since the RHS is nontrivial it will have $\boxed{\text{no solutions}}$.

Ex:

$$\begin{matrix} \\ 3 \\ 2 \end{matrix}\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 3 & 1 & -2 & | & 5 \\ 2 & 2 & 1 & | & 4 \end{bmatrix} = \begin{matrix} \\ \\ 2 \end{matrix}\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 2 & 7 & | & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & -7 & | & -14 \end{bmatrix}$$

then $x_3 = 2$, $x_2 = -3$, $x_1 = 4$.

Ex:

$$\begin{matrix} \\ 2 \\ 4 \end{matrix}\begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 4 & -3 & 7 & | & 5 \\ 8 & -9 & 15 & | & 10 \end{bmatrix} = \begin{matrix} \\ \\ 3 \end{matrix}\begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 0 & -3 & 1 & | & -1 \\ 0 & -9 & 3 & | & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 0 & -3 & 1 & | & -1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Clearly this matrix is singular, and since the RHS is nontrivial it will have $\boxed{\text{no solutions}}$.