Sec. 1.5 Matrix Inverses

In linear algebra we want to find a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. We can only find an inverse if A is nonsingular, so invertible and nonsingular mean the same thing, and noninvertible and singular mean the same thing. If A can be put into upper triangular form, it is nonsingular, otherwise it is singular.

Note: nonsquare matrices do not have inverses, but we can have either a right or left hand inverse, which we will talk about later.

Now lets do an example where we find the inverse of a matrix. Once more we use our usual 3×3 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \tag{1}$$

We append the identity matrix to A and use Gauss-Jordan elimination.

$$\begin{matrix} \\ 2 \\ -1 \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] = \begin{matrix} \\ \\ -1 \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right] = \begin{matrix} 1 \\ -2 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$= \begin{matrix} -1/8 \\ \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \begin{matrix} 1/2 \\ -1/8 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 12/8 & -5/8 & -6/8 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 12/16 & -5/16 & -6/16 \\ 0 & 1 & 0 & 4/8 & -3/8 & -2/8 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 12/16 & -5/16 & -6/16 \\ 4/8 & -3/8 & -2/8 \\ -1 & 1 & 1 \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

This one is easy since there are only diagonal terms, so

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Ex: For this one

$$\begin{matrix} \\ 3 \end{matrix}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] = \begin{matrix} 2 \\ \\ \end{matrix}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

Don't be fooled by the simplicity of this answer though. Even for 2×2, the pattern may not be as you see here. This one is a special case since the determinant (which we will cover later) is 1.

Ex: Similarly,

$$\begin{matrix} \\ 3 \\ -2 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] = \begin{matrix} \\ \\ -1 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -6 & -3 & 1 & 0 \\ 0 & 2 & 7 & 2 & 0 & 1 \end{array}\right] = \begin{matrix} 2 \\ -6 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -6 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$= \begin{matrix} -1/2 \\ \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -2 \\ 0 & -2 & 0 & -9 & 7 & 6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \begin{matrix} \\ -1/2 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 3/2 & 1 \\ 0 & -2 & 0 & -9 & 7 & 6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 3/2 & 1 \\ 0 & 1 & 0 & 9/2 & -7/2 & -6/2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Properties:

(1) $\left(A^{-1}\right)^{-1}$,

(2) $\left(A^{k}\right)^{-1} = \left(A^{-1}\right)^{-1}$

(3) $(cA)^{-1} = \dfrac{1}{c}A^{-1}$

(4) $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$

(5) $(AB)^{-1} = B^{-1}A^{-1}$