Sec. 2.1 and 2.2 Vector spaces and subspaces

Here we saw some definitions and how they apply to some examples.

Definition 1. The vectors $v_1, \dots, v_n$ are said to be Linearly Independent if $c_1v_1+\cdots+c_nv_n \neq 0$ whenever $c_i \neq 0$ for $i = 1, \dots, n$; otherwise they are said to be Linearly Dependent.

Definition 2. The expression $c_1v_1 + \cdots + c_nv_n$ is said to be a Linear Combination of $v_1, \dots, v_n$.

A vector space is simply a space that contains all of the axioms of vector addition and scalar multiplication, and is self-contained; i.e., addition and scalar multiplication of any combination of vectors will produce a vector in that space.

Definition 3. A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space: Linear combinations stay in the subset;

(i) if we add any vectors x and y in the subspace $x + y$ is in the subspace,
(ii) if we multiply any vector x in the subspace by any scalar c, cx is in the subspace.

Now lets look at some examples

Ex: $W \subseteq \mathbb{R}^4$ such that $\forall \quad x, y, z \in \mathbb{R}$,

$$\begin{vmatrix} x \\ y \\ z \\ 0 \end{vmatrix} \in W.$$

W is clearly nonempty and a subset of V. We just have to check the properties listed in Def. 3.

(i)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ 0 \end{bmatrix}$$

By the axioms of arithmetic $x_i + y_i$ will be real numbers, and the last entry is zero, so this is in W.

(ii)

$$c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ 0 \end{bmatrix}$$

Again, by the axioms of arithmetic cx_i will be real numbers, and the last entry is zero, so this is in W as well.
Since both properties are satisfied, W is a subspace of $\mathbb{R}^4$.

Ex: $V \subseteq \mathbb{R}^3$ such that $\forall \quad x, y \in \mathbb{R}$,

$$\begin{vmatrix} x \\ y \\ 4x - 5y \end{vmatrix} \in V.$$

Just as the previous problem

(i)

$$\begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - 5y_1 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 4(x_1 + x_2) - 5(y_1 + y_2) \end{bmatrix}$$

By the axioms of arithmetic all three entries will be real, thus matching the definition of the set V, and hence the vector is in V.

(ii)

$$c \begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - 5y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ 4cx_1 - 5cy_1 \end{bmatrix} \in V$$

So, this too satisfies our conditions.
Therefore, V is a subspace in $\mathbb{R}^3$.

Ex: $W \subseteq \mathbb{R}^3$ such that $\forall \quad x, y \in \mathbb{R}$,

$$\begin{vmatrix} x \\ y \\ -1 \end{vmatrix} \in W$$

Here both properties can be violated. For property (ii),

$$c \begin{bmatrix} x \\ y \\ -1 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ -c \end{bmatrix}$$

In general $-c \neq -1$, so this vector cannot be in W. Hence, W is not a subspace of $\mathbb{R}^3$.

Ex: $V \subseteq \mathbb{R}^2$ such that $\forall \quad x, y \in \mathbb{R}$,

$$\begin{vmatrix} x \\ y \end{vmatrix} \in V.$$

Here only property (ii) is violated,

$$\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \notin Q,$$

therefore is not in V, and V is not a subspace of $\mathbb{R}^2$.