8.3 Diagonalization

Suppose $A_{n\times n}$ has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then A has a factorization such that $S^{-1}AS = \Lambda$ is a diagonal matrix where the eigenvalues of A are on the diagonal. We can see this in the following calculation,

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = S\Lambda \Rightarrow A = S\Lambda S^{-1}$$

A matrix is not diagonalizable if the eigenvectors are not linearly independent.

Caveat: there is no direct connection between this and invertibility. Diagonalizability depends on the linear independence of eigenvectors and invertibility depends on the linear independence of the columns of a matrix.

Now lets look at some examples.

Ex: Find S and Λ in the $A = S\Lambda S^{-1}$ factorization of

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Solution: First we find the eigenvalues

$$\begin{vmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{vmatrix} = \frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} = \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0, 1$$

Next we find the eigenvectors

$$\begin{pmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{pmatrix} x = 0 \Rightarrow x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Ex: Find S and Λ in the $K = S\Lambda S^{-1}$ factorization of

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

First we find the eigenvalues,

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i.$$

Next we find the eignevectors

$$\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} x = 0 \Rightarrow x = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Then

$$\Lambda = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

Lets look at a couple of interesting results that may be useful to us.

Theorem 1. *The eigenvalues of A^2 is $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ if the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

Proof.

$$A^2 x = A(Ax) = A(\lambda x) = \lambda A x = \lambda^2 x.$$

□

Alternate proof.

$$A = S\Lambda S^{-1} \Rightarrow A^2 = \left(S\Lambda S^{-1}\right)\left(S\Lambda S^{-1}\right) = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}.$$

□

We can extend this to higher powers and inverses; i.e., $A^k = S\Lambda^k S^{-1}$ and $A^{-1} = S\Lambda^{-1}S^{-1}$ if $\lambda_1, \ldots, \lambda_n \neq 0$.