Ch. 1 Linear Equations

Linear equations are equations that create a line. One main goal of linear algebra is to find where lines intersect in many dimensions (e.g., millions of dimensions). Lets look at some examples.

Ex:　$x + y = 1$ and $x - y = 0$.
　　　Solution: $x = y \Rightarrow 2x = 1 \Rightarrow \boxed{x = y = 1/2}$.
Ex:　$x + 2y = 3$ and $4x + 5y = 6$.
　　　Solution: $x = 3 - 2y \Rightarrow 4x + 5y = 12 - 8y + 5y = 12 - 3y = 6$, so $\boxed{y = 2} \Rightarrow \boxed{x = \text{-}1}$.

We learned to solve linear equations in this manner in high school. Perhaps we even learned how to isolate terms through multiplying and adding equations. However, these methods would not work on a computer, which is what we need for say a million equations with a million unknowns. Lets think about how a computer might be able to solve these problems.

Ex:　$x + 2y = 3$ and $4x + 5y = 6$.
　　　Solution:

$$\begin{array}{c} x + 2y = 3 \\ (4x + 5y = 6) - \textcolor{red}{4(x + 2y = 3)} \end{array} \Rightarrow \begin{array}{c} x + 2y = 3 \\ 0 - 3y = -6 \end{array} \Rightarrow \boxed{y = 2} \Rightarrow x + 4 = 3 \Rightarrow \boxed{x = \text{-}1}$$

This is called <u>Gaussian Elimination</u>.

This seems dumb for only two equations, but for a million by million your computer can probably do this in less than a second. In order to tell the computer to conduct this operation, you must first know how to do it yourself.

Sometimes a system of equations can have **no solution** or **infinitely many solutions**. If it has **infinitely many solutions** the system is called a <u>homogeneous system</u>, and if it has **no solutions** it is called a <u>singular system</u>. However, as we will see later, a matrix will be called singular if the final system has infinitely many solutions or no solutions.

Ex:　$x + y = 1$, $2x + 2y = 2$.
　　　Notice that an infinite number of x and y combinations (e.g., $x = 1, y = 0$; $x = 0, y = 1$; $x = 1/2, y = 1/2$; etc)
Ex:　$x + 2y = 3$, $4x + 8y = 6$.
　　　Notice that Eq1 - Eq2 gives us $0 = -6$, and therefore does not have a solution.

2.1 Notation

<u>Coefficient matrix</u>
Consider

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \tag{1}$$

Then the matrix

$$A_{3\times 3} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

is called the <u>coefficient matrix</u> of (1). Also, the matrix A is said to be a 3×3 matrix as it has 3 rows and 3 columns.

<u>Addition</u>
Matrix addition works just like scalar addition, and we just add the respective elements together.
Consider

$$B_{3\times 2} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}, C_{3\times 2} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$B + C = \begin{bmatrix} 3 & 3 \\ 0 & 1 \\ 1 & 6 \end{bmatrix}$$

Further, the matrices B and C are of size 3×2 since they have 3 rows and 2 columns

Scalar multiplication

Just multiply the scalar term with each element,

Ex:
$$2B = \begin{bmatrix} 4 & 2 \\ 6 & 0 \\ 0 & 8 \end{bmatrix}$$

Ex:
$$2C = \begin{bmatrix} 2 & 4 \\ -6 & 2 \\ 2 & 4 \end{bmatrix}$$

Ex:
$$2(B+C) = 2(C+B) = \begin{bmatrix} 6 & 6 \\ 0 & 2 \\ 2 & 12 \end{bmatrix}$$

Transpose

We denote transpose with a T superscripted onto the matrix,

Ex:
$$C^T = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Ex:
$$B^T = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Dot (inner) product

Most of you have seen dot products in your Calculus courses. We will see why this is called an inner product as well in later sections.

Ex:
$$\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 1 \cdot 2 + (-3) \cdot 3 + 1 \cdot 0 = -7.$$

Size

The size of a matrix is written as the number of rows by the number of columns, and if shown is given as a subscript.

Multiplication

We can only multiply matrices if the number of rows of the first is equivalent to the number of columns of the second. In order to multiply we will do the dot product of the row of the first matrix with the column of the second. Here is a generalization of that process for a 2×2 and 3×3, and it is a simple extension to other sizes.

2×2:
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \tag{2}$$

3×3:
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a13b_{23} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix} \tag{3}$$

Ex:
$$BC^T = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 4 & -5 & 4 \\ 3 & -9 & 3 \\ 8 & 4 & 8 \end{bmatrix}$$

Ex:
$$CB^T = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Notice that $BC^T = (CB^T)^T$ and vise-versa.

Ex:
$$C^T B = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{bmatrix} -7 & 5 \\ 7 & 10 \end{bmatrix}$$

Ex:

$$B^T C = \begin{bmatrix} -7 & 7 \\ 5 & 10 \end{bmatrix}$$

Notice that now that we know how to transpose products, we did not have to do any work since $B^T C = (C^T B)^T$.

Matrix form of linear systems

Now that we know how to multiply matrices we may convert our system of equations into a matrix equation, which the computer can understand. Lets again consider (1),

$$(1) \Rightarrow Ax = b \Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \tag{4}$$

We talked about a lot of notation, but we skipped some intentionally. We'll get back to those later.

We saw that matrices of different sizes cannot be commuted; i.e., $A_{2\times3}B_{3\times2} \neq B_{3\times2}A_{2\times3}$. However, with scalar multiplication we can commute (e.g., $5 \cdot 2 = 2 \cdot 5$), so can two matrices of the same size commute; i..e, $A_{2\times2}B_{2\times2} \overset{?}{=} B_{2\times2}A_{2\times2}$.