

5.1-5.2 DOT PRODUCT REVIEW AND INNER PRODUCT SPACES

We reviewed how to do dot products. Then we talked about orthogonality and projections.

Orthogonality

We notice that right angles are the most important angles in linear algebra. Recall that the four fundamental subspaces meet at right angles. Also our standard bases are produced using orthogonal vectors, in fact they are even better; they are unit vectors, which is referred to as normal. So, $\hat{i}, \hat{j}, \hat{k}$ are orthonormal.

Ex: The following vectors are orthogonal.

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 0.$$

Ex: Same as above

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0.$$

Theorem 1. *If nonzero vectors v_1, \dots, v_k are mutually orthogonal (every vector is perpendicular to every other vector), then those vectors are linearly independent.*

There are times when orthogonal will not mean exactly right angles, such as when we look at functions:

$$\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

Notice that these vectors are also normal since $\cos^2 \theta + \sin^2 \theta = 1$.

Definition 1. A basis is an orthonormal basis if it consists of mutually orthogonal unit vectors; i.e., perpendicular vectors of length one.

Also, we notice that we can only have certain combinations of orthogonal vectors in a finite subspace. Take \mathbb{R}^3 for example. We can only have two lines or a line and a plane, but we cannot have two planes orthogonal to each other.

Theorem 2 (orthogonality). *The row space $\mathcal{C}(A^T)$ and the nullspace $\mathcal{N}(A)$ are orthogonal to each other, as are the column space $\mathcal{C}(A)$ and the left nullspace $\mathcal{N}(A^T)$.*

Proof. Notice that $Ax = 0$ for $\mathcal{N}(A)$, but the nonzero rows of A make up $\mathcal{C}(A^T)$. So, each $x \in \mathcal{N}(A)$ is orthogonal to each row of A . □

Projections onto lines

The shortest distance from a point \vec{b} onto a line through \vec{a} is via a line through \vec{b} that is perpendicular to the line through \vec{a} . This will meet the line through \vec{a} at a point \vec{p} . Then \vec{p} is simply a scaled version of \vec{a} , so $\vec{p} = \hat{x}\vec{a}$, and the perpendicular line is $\vec{b} - \vec{p}$. Then $\vec{a}^T(\vec{b} - \hat{x}\vec{a}) = 0$, and solving for \hat{x} give us $\hat{x} = (\vec{a}^T\vec{b})/(\vec{a}^T\vec{a})$.

Definition 2. The projection of the vector \vec{b} onto the line in the direction of \vec{a} is

$$\vec{p} = \hat{x}\vec{a} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} \right) \vec{a}. \tag{1}$$

While this leads us to the Cauchy-Schwartz identity, since we didn't talk about it much we shall skip it here.

Ex: Project $b = (1, 2, 3)$ onto the line through $a = (1, 1, 1)$ to get \hat{x} and p .

Solution:

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{6}{3} = 2 \Rightarrow p = \hat{x}a = (2, 2, 2).$$

Now lets look at a couple of problems from the book on page 252 # 73

75) (a) For u onto v we do

$$\hat{x} = \frac{v^T u}{u^T u} = \frac{8}{2} = 4 \Rightarrow p = \hat{x}v = (4, -4, 0)$$

(b) For v onto u we do

$$\hat{x} = \frac{u^T v}{u^T u} = \frac{8}{7} \Rightarrow p = \hat{x}u = \left(\frac{8}{7}, -\frac{24}{35}, \frac{8}{35} \right)$$