Last time we asked if matrices of the same size can commute. Lets look at a simple example.

Ex: Consider

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

Notice

$$AB = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \neq BA = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

2.2 Properties of Matrix Operations

Matrix addition works just like scalar addition.

Matrix multiplication: $(AB)C = A(BC)$, $A(B + C) = AB + AC$, $AB \neq BA$. So, in general matrices do not commute. Is there a matrix that commutes with everything?

Consider the 2×2 matrix

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \tag{1}$$

and a generic 2×2 matrix

$$A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix},$$

then $AI = A = IA$. This is multiplicative identity of matrices, and is called the identity matrix. For a general $n \times n$ matrix it takes the form,

$$I_n = \begin{bmatrix}1 & & & \\ & 1 & & \text{\Large 0} \\ & & \ddots & \\ & \text{\Large 0} & & \ddots \\ & & & & 1\end{bmatrix} \tag{2}$$

that is, ones down the diagonal and zeros everywhere else, so for a 3×3 matrix it would be

$$I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Now lets look at properties of the transpose: $(A^T)^T = A$, $(A + B)^T = A^T + B^T$, $(cA)^T = c(A^T)$, $(AB)^T = B^T A^T$.

Notice that a dot product of two vectors can also be written as $v\dot{w} = v^T w$.

1.2 Gaussian Elimination

Now lets go back and do a bunch of Gaussian Elimination problems. Again consider our equation from last time

$$\begin{aligned}2u + v + w &= 5\\ 4u - 6v &= -2\\ -2u + 7v + 2w &= 9\end{aligned} \tag{3}$$

we will write this as an augmented matrix by appending the right hand side (RHS) to the coefficient matrix,

$$\begin{matrix}2\\ \\ \end{matrix}\begin{bmatrix}2 & 1 & 1 & | & 5\\4 & -6 & 0 & | & -2\\-2 & 7 & 2 & | & 9\end{bmatrix} = \begin{matrix}\\ \\ -1\end{matrix}\begin{bmatrix}2 & 1 & 1 & | & 5\\0 & -8 & -2 & | & -12\\-2 & 7 & 2 & | & 9\end{bmatrix} = \begin{matrix}\\ \\ -1\end{matrix}\begin{bmatrix}2 & 1 & 1 & | & 5\\0 & -8 & -2 & | & -12\\0 & 8 & 3 & | & 14\end{bmatrix} = \begin{bmatrix}2 & 1 & 1 & | & 5\\0 & -8 & -2 & | & -12\\0 & 0 & 1 & | & 2\end{bmatrix}$$

This means $\boxed{w = 2}$, then we plug into the second equation to get $\boxed{v = 1}$, and finally the first to get $\boxed{u = 1}$.

The elements down the diagonal are called pivots. The augmented matrix is said to be in row-echelon form. Th original matrix,

$$\begin{bmatrix}2 & 1 & 1\\0 & -8 & -2\\0 & 0 & 1\end{bmatrix}$$

is said to be in upper triangular form.

Here are some problems we did from the book pp 22 - 23.

25)

$$\begin{matrix} \\ 3 \end{matrix}\begin{bmatrix} 1 & 3 & | & 11 \\ 3 & 1 & | & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & | & 11 \\ 0 & -8 & | & -12 \end{bmatrix} \Rightarrow \boxed{\text{y} = 3} \Rightarrow \boxed{\text{x} = 2};$$

27)

$$\begin{matrix} \\ -2 \end{matrix}\begin{bmatrix} -1 & 2 & | & 3/2 \\ 2 & -4 & | & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & | & 3/2 \\ 0 & 0 & | & -6 \end{bmatrix}$$

Clearly this matrix is singular, and since the RHS is nontrivial it will have $\boxed{\text{no solutions}}$.

31)

$$\begin{matrix} \\ 3 \\ 2 \end{matrix}\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 3 & 1 & -2 & | & 5 \\ 2 & 2 & 1 & | & 4 \end{bmatrix} = \begin{matrix} \\ \\ 2 \end{matrix}\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 2 & 7 & | & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & -7 & | & -14 \end{bmatrix}$$

then $x_3 = 2$, $x_2 = -3$, $x_1 = 4$.

33)

$$\begin{matrix} \\ 2 \\ 4 \end{matrix}\begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 4 & -3 & 7 & | & 5 \\ 8 & -9 & 15 & | & 10 \end{bmatrix} = \begin{matrix} \\ \\ 3 \end{matrix}\begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 0 & -3 & 1 & | & -1 \\ 0 & -9 & 3 & | & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & | & 3 \\ 0 & -3 & 1 & | & -1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Clearly this matrix is singular, and since the RHS is nontrivial it will have $\boxed{\text{no solutions}}$.