4.1 - 4.3 Vector spaces and subspaces

Here we saw some definitions and how they apply to some examples.

Definition 1. The vectors $v_1, \dots, v_n$ are said to be Linearly Independent if $c_1v_1 + \cdots + c_nv_n \neq 0$ when $c_i \neq 0$ for $i = 1, \dots, n$; otherwise they are said to be Linearly Dependent.

Definition 2. The expression $c_1v_1 + \cdots + c_nv_n$ is said to be a Linear Combination of $v_1, \dots, v_n$.

A vector space is simply a space that contains all of the axioms of vector addition and scalar multiplication, and is self-contained; i.e., addition and scalar multiplication of any combination of vectors will produce a vector in that space.

Definition 3. A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space: Linear combinations stay in the subset;

(i) if we add any vectors x and y in the subspace $x + y$ is in the subspace,
(ii) if we multiply any vector x in the subspace by any scalar c, cx is in the subspace.

Now lets do some problems from pg. 173.

1) W is clearly nonempty and a subset of V. We just have to check the properties listed in Def. 3.

(i)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ 0 \end{bmatrix}$$
By the axioms of arithmetic $x_i + y_i$ will be real numbers, and the last entry is zero, so this is in W.

(ii)
$$c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ 0 \end{bmatrix}$$
Again, by the axioms of arithmetic cx_i will be real numbers, and the last entry is zero, so this is in W as well.

Since both properties are satisfied, W is a subspace of V.

2) Just as the previous problem

(i)
$$\begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - 5y_1 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 4(x_1+x_2) - 5(y_1+y_2) \end{bmatrix}$$
By the axioms of arithmetic all three entries will be real, thus matching the definition of the set W, and hence the vector is in W.

(ii)
$$c\begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - 5y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ 4cx_1 - 5cy_1 \end{bmatrix}$$

7) Here both properties can be violated. For property (ii),
$$c\begin{bmatrix} x \\ y \\ -1 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ -c \end{bmatrix}$$
In general $-c \neq -1$, so this vector cannot be in W. Hence, W is not a subspace of V.

9) Here only property (ii) is violated,
$$\sqrt{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \notin Q,$$
therefore is not in W, and W is not a subspace of V.