4.4 - 4.5 Spanning sets and linear independence; basis and dimension.

First we notice that n vectors cannot be linearly independent in $\mathbb{R}^m$ if $n > m$. Further, if we do not have enough vectors a linear combination will not be able to create any other vector in the space. Lets see what this means about the dimension of the space.

Definition 1. If a vector space V consists of all linear combinations of $w_1, \ldots, w_n$, then these vectors span the space. Every vector $v \in V$ is some combination of $w's$; i.e., $v = c_1 w_1 + \cdots + c_n w_n$.

For example

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

spans the $x - y$ plane in $\mathbb{R}^3$. However, we notice that they are not linearly independent since $w_3 = -2w_1$.

Definition 2. A basis for V is a sequence of vectors having the following two properties:

(1) The vectors are linearly independent (not too many vectors)
(2) They span the space V (not too few vectors)

We sketched this in class. Make sure you understand that picture.

Definition 3. Any two bases for a vector space V contains the same number of vectors. This number, which is shared by all bases and expresses the number of "degrees of freedom" of the space, is the dimension of V.

This leads us to a coupe of theorems.

Theorem 1. *If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$.*

Theorem 2. *Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary. Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.*

Now lets look at some examples on pg. 184

1) Do check if vectors are linear combinations, we just assume they are using c_1 and c_2, then we check if c_1 and c_2 are nontrivial.

a)

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2c_1 + 5c_2 \\ -c_1 + 0 \\ 3c_1 + 4c_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

This gives us $c_1 = 2$, $c_2 = -1$ and the other equation is satisfied. So,

$$z = 2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

c) We can just use the addition from above

$$\begin{bmatrix} 2c_1 + 5c_2 \\ -c_1 + 0 \\ 3c_1 + 4c_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ -8 \\ 12 \end{bmatrix}$$

Then $c_1 = 8$, $c_2 = -3$, and the other equation is satisfied, so

$$w = 8 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

d) We do the same as the last two

$$\begin{bmatrix} 2c_1 + 5c_2 \\ -c_1 + 0 \\ 3c_1 + 4c_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then plugging into the second equation gives us $c_1 = -1$, and the first gives $c_2 = 3/5$, but $3c_1 + 4c_2 \neq 1$, so we cannot write u as a linear combination of the vectors in S.

9) S does span $\mathbb{R}^2$, which we showed graphically, but also if it didn't, then s_1 would be a multiple of s_2, but if $s_1 = cs_2 \Rightarrow c = 0$.

11) Same reasoning as (9).

19) S spans $\mathbb{R}^3$. We saw that we can write x_1, x_2, x_3 independently as functions of c_1, c_2, c_3, but I showed a better way in the next section.

21) S does not span $\mathbb{R}^3$. Not enough vectors, but it does span the plane.

27) These are linearly independent since

$$c_1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2c_1 + 3c_2 \\ 2c_1 + 5c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_2 = 2c_1/3 \text{ and } c_2 = -2c_1/5 \Rightarrow c_1 = c_2 = 0.$$

29) Any set with the zero vector is linearly dependent.

Now lets look at some problems from pg. 193

1) The vectors will be $e_1, \ldots, e_6$ where e_i has a 1 in the i^{th} entry and zero elsewhere.

7) Linearly dependent because of the zero vector.

9) Not enough vectors.

21) Linearly dependent.

39) Linearly independent and spans the space, so it does form a basis. Notice that it does have enough vectors and we can show it is linearly independent graphically or by solving

$$c_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4c_1 + 5c_2 \\ -3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = -5c_2/4 \text{ and } c_1 = 2c_2/3 \Rightarrow c_1 = c_2 = 0.$$

41) Same as the previous problem and

$$c_1 \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 \\ 5c_1 + c_2 \\ 3c_1 + 2c_2 + 6c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$