Skipped problems from first exam

Fall 2007 4a) We first convert this into standard form:

$$y' + \frac{\ln(t+1)}{(t-3)(t+3)}y = \frac{2}{(t-3)(t+3)\cos t}.$$

The points of discontinuity are: $t = 3, -3, -1, \pi/2 \pm n\pi$. Since we need to include the initial point, $t = 0$, our interval of existence is $(-1, \pi/2)$.

Spring 2007 1a) We convert to standard form:

$$y'' + \frac{e^t}{(t+2)(t-4)}y' + \frac{\ln t}{(t+2)(t-4)}y = \frac{t^2}{(t+2)(t-4)}.$$

The points of discontinuity are: $t = -2, 4, 0$. Since we need to include the initial point, our interval of existence is $(0, 4)$.

Spring 2012 6b) We take the Wronskian,

$$W = \begin{vmatrix} t^{-3} & y_2 \\ -3t^{-4} & y_2' \end{vmatrix} = t^{-3}y_2' + 3t^{-4}y_2 = 4t^3 \Rightarrow y_2' + 3t^{-1}y_2 = 4t^6.$$

Then,

$$\mu = \exp\left(3\int^t \frac{d\tau}{\tau}\right) = t^3 \Rightarrow t^3 y_2 = \int^t 4\tau^9 d\tau = \frac{2}{5}t^{10} \Rightarrow y_2 = \frac{2}{5}t^7.$$

Spring 2012 6c) Yes, because the Wronskian, $W(y_1, y_2) \neq 0$.

Fall 2007 Solutions

(1) (a) (i) Because of the absolute value, we must split up the problem,

$$t-1>0: W = \begin{vmatrix} t-1 & 2(t-1) \\ 1 & 2 \end{vmatrix} = 2(t-1)-2(t-1)=0$$

$$t-1<0: W = \begin{vmatrix} 1-t & 2(t-1) \\ -1 & 2 \end{vmatrix} = -2(t-1)+2(t-1)=0$$

Therefore, the solutions are not linearly independent.

(ii) Taking the Wronskian gives,

$$W = \begin{vmatrix} 3t+1 & t+3 \\ 3 & 1 \end{vmatrix} = 3t+1-3t-9=-8 =\neq 0.$$

Therefore, they are linearly independent.

(b) The Wronskian gives,

$$W = \begin{vmatrix} x & g \\ 1 & g' \end{vmatrix} = xg'-g = x \Rightarrow g' - g/x = 1.$$

Then,

$$\mu = \exp\left(-\int^x \frac{d\xi}{\xi}\right) = \frac{1}{x} \Rightarrow \frac{g}{x} = \int^x \frac{1}{\xi}d\xi = \ln x \Rightarrow g = x\ln x.$$

(2) (a) We find the homogeneous solution, $r^2 - r = 0 \Rightarrow r = 0, 1$, then $y_c = c_1 + c_2e^t$. Our particular solution is,

$$y_p = Ate^t + Bt^2 + Ct \Rightarrow y_p' = Ate^t + Ae^t + 2Bt + C \Rightarrow y_p'' = Ate^t + 2Ae^t + 2B.$$

Plugging this into the ODE gives,

$$Ate^t + 2Ae^t + 2B - Ate^t - Ae^t - 2Bt - C = 2e^t - 1 - t \Rightarrow A = 2, B = \frac{1}{2}, C = 2.$$

Then the particular solution is,

$$y_p = 2te^t + \frac{1}{2}t^2 + 2t.$$

(b) The general solution is,

$$y = c_1 + c_2e^t + 2te^t + \frac{1}{2}t^2 + 2t.$$

(3) (a) Let $y = u(x)e^{-x}$.

Short cut: First convert the equation into standard form,

$$y'' + \frac{x-1}{x}y' - \frac{1}{x}y = 0.$$

Now use the formula we derived in class:

$$y_1u'' + (2y_1' + p(x)y_1)u' = e^{-x}u'' + \left(-2e^{-x} + \frac{x-1}{x}e^{-x}\right)u' = 0 \Rightarrow u'' - \frac{x+1}{x}u' = 0.$$

Not short cut: $y' = u'e^{-x} - ue^{-x}$ and $y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}$. Plugging into the ODE gives,

$$u''xe^{-x} - 2u'xe^{-x} + \cancel{uxe^{-x}} + u'xe^{-x} \cancel{-uxe^{-x}} - u'e^{-x} + \cancel{ue^{-x}} \cancel{-ue^{-x}} = u''xe^{-x} - u'xe^{-x} - u'e^{-x} = 0$$

$$\Rightarrow xu'' - (x+1)u' = 0 \Rightarrow u'' - \frac{x+1}{x}u' = 0.$$

Using either way you will get the equation $u'' - [(x+1)/x]u' = 0$. Now, let $v = u'$, then,

$$v' = \frac{x+1}{x}v \Rightarrow \int \frac{dv}{v} = \int \left(1 + \frac{1}{x}\right) dx \Rightarrow \ln v = x + \ln x + C_0 \Rightarrow v = kxe^x$$

$$\Rightarrow u = k \int xe^x dx = k(x-1)e^x + C_1 \Rightarrow y = k(x-1) + C_1e^{-x}.$$

Then, $y_2 = x - 1$.

(b) Notice here we can extract the roots: $r = 1, 1, -1+2i, -1-2i$, then $(r-1)^2(r+1+2i)(r+1-2i) = r^4 + 2r^2 - 8r + 5$, then our equation is,

$$y^{(4)} + 2y'' - 8y' + 5y = 0.$$

(4) To make life easier I am going to divide through by 2,

$$y'' + 2y' + y = \frac{1}{2t}e^{-t} \Rightarrow r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow y_c = c_1e^{-t} + c_2te^{-t}.$$

So, $y_1 = e^{-t}$ and $y_2 = te^{-t}$. Recall, for variation of parameters we set $y_p = u(t)y_c$ and plug it into the ODE, which gives

$$y = -y_1 \int \frac{y_2 f(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f(t)}{W(y_1, y_2)} dt.$$

Now we take the Wronskian,

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}.$$

Calculating the individual integrals gives,

$$\int \frac{y_2 f(t)}{W(y_1,y_2)} dt = \int \frac{(\cancel{t}\cancel{e^{-t}})\,(\cancel{e^{-t}}/2\cancel{t})}{\cancel{e^{-2t}}} dt = \frac{1}{2}t + c_3.$$

$$\int \frac{y_1 f(t)}{W(y_1,y_2)} dt = \int \frac{(\cancel{e^{-t}})\,(\cancel{e^{-t}}/2t)}{\cancel{e^{-2t}}} dt = \frac{1}{2}\ln t + c_4.$$

Then we get,

$$y = -\frac{1}{2}te^{-t} - c_3e^{-t} + \frac{1}{2}te^{-t}\ln t + c_4te^{-t} = c_5te^{-t} - c_3e^{-t} + \frac{1}{2}te^{-t}\ln t.$$

Then the particular solution is,

$$y_p = \frac{1}{2}te^{-t}\ln t.$$

(5) We first find the homogeneous solution,

$$r^4+2r^3+2r^2 = r^2(r^2+2r+2) = 0 \Rightarrow r = \frac{1}{2}(-2\pm\sqrt{-4}) = -1\pm i \Rightarrow y_c = c_1+c_2t+e^{-t}(c_3\cos t+c_4\sin t).$$

Then, the particular solution is,

$$y_p = (At+B)e^{-t} + te^{-t}(D\cos t + E\sin t) + Fe^t.$$

(6) We find the roots,

$$r^3-r^2-r+1 = r^2(r-1)-(r-1) = (r-1)(r^2-1) = (r+1)(r-1)^2 = 0 \Rightarrow r = -1, 1.$$

Then we get the general solution,

$$y = (c_1+c_2t)e^t+c_3e^{-t} \Rightarrow y' = c_2e^t+(c_1+c_2t)e^t-c_3e^{-t} \Rightarrow y'' = 2c_2e^t+(c_1+c_2t)e^t+c_3e^{-t}.$$

Now we plug in the initial conditions, $y(0) = c_1 + c_3 = 2 \Rightarrow c_3 = 2 - c_1$, $y'(0) = c_1 + c_2 - c_3 = -1 \Rightarrow 2c_1 + c_2 = 1$, and $y''(0) = 2c_2 + c_1 + c_3 = 0 \Rightarrow 2c_2 = -2$, then we get $c_2 = -1$, $c_1 = 1$, $c_3 = 1$, then our solution is,

$$y = (1-t)e^t - e^{-t}.$$

Spring 2012 Solutions

(1) (a) We have

$$r^2 + 2r + 17 = 0 \Rightarrow r = \frac{1}{2}(-2 \pm \sqrt{-4 - 4 \cdot 17}) = -1 \pm \sqrt{-16} = -1 \pm i4.$$

This gives us a general solution of,

$$y = e^{-t}(A \cos 4t + B \sin 4t).$$

(b) We have,

$$4r^2 + 4r + 1 = (2r+1)^2 = 0 \Rightarrow y = (c_1 + c_2 t)e^{-t/2}.$$

(2) It's easy to verify that this is a solution. For the other solution let $y = vy_1$. Yes, I know I used "u" for the 2007 problem, just go with it, it's the same thing.

Short cut: We need to convert into standard form,

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 0.$$

Now we employ our formula,

$$y_1 v'' + (2y_1' + py_1)v' = tv'' + (2+1)v' \Rightarrow v'' = -\frac{3}{t}v'.$$

Not short cut: $y' = tv' + v$ and $y'' = tv'' + 2v'$, then

$$t^3 v'' + 2t^2 v' + t^2 v' + \cancel{tv - vt} = t^3 v'' + 3t^2 v' = 0 \Rightarrow v'' = -\frac{3}{t}v'.$$

In the end we get the same equation. We let $u = v'$,

$$u' = -\frac{3}{t}u \Rightarrow \int \frac{du}{u} = -\int \frac{3}{t}dt \Rightarrow \ln u = -3\ln t + C_0 \Rightarrow u = k_0 t^{-3}$$

$$\Rightarrow v = -\frac{k_0}{2}t^{-2} + C_1 \Rightarrow y = k_1 \frac{1}{t} + C_1 t \Rightarrow y_2 = \frac{1}{t}.$$

(3) (a) Since there is no damping we have the model, $mx'' + kx = 0$. To find k we have $k = F/x = 8\text{lb}/2\text{ft} = 4\text{lb/ft}$. And the mass is $m = 8\text{lb}/32\text{ft/s}^2 = 1/4\text{lb} \cdot \text{s}^2/\text{ft}$. Then our equation becomes, $x'' + 16x = 0$. And we have $x(0) = 1$, $x'(0) = -4$ where down is positive and up is negative, i.e. with gravity and against gravity. We get the general solution,

$$r^2 + 16 = 0 \Rightarrow r = \pm i4 \Rightarrow x = A\cos 4t + B \sin 4t.$$

The initial conditions give, $x(0) = A = 1$, $x'(0) = 4B = -4 \Rightarrow B = -1$. Then our solution is,

$$x = \cos 4t - \sin 4t.$$

(b) $R = \sqrt{A^2 + B^2} = \sqrt{2}$ and $\tan \delta = B/A = -1/1 \Rightarrow \delta = -\pi/4$.

(4) We get a general solution of,

$$r^2 - 2r + 17 = 0 \Rightarrow r = \frac{1}{2}(2 \pm 2\sqrt{-16}) = 1 \pm i4 \Rightarrow y = e^t(A\cos 4t + B\sin 4t).$$

From the first initial condition we get $y(\pi/4) = -A\exp(\pi/4) = 1 \Rightarrow A = -\exp(-\pi/4)$. For the second initial condition, lets first take the derivative,

$$y' = e^t(A\cos 4t + B\sin 4t) + e^t(-4A\sin 4t + 4B\cos 4t).$$

Then, $y'(\pi/4) = -Ae^{\pi/4} - 4Be^{\pi/4} = 1 - 4Be^{\pi/4} = -1 \Rightarrow B = e^{-\pi/4}/2$. So our solution is,

$$y = e^{t-\pi/4}\left(\frac{1}{2}e^{-\pi/4}\sin 4t - e^{-\pi/4}\cos 4t\right).$$

(5) (a) We get a general solution of,

$$r^2 - 2r + 2 = 0 \Rightarrow r = \frac{1}{2}(2 \pm i2) = 1 \pm i \Rightarrow y = e^t(A\cos t + B\sin t).$$

For the fundamental set we have,

$$y_1(0) = A = 1,\ y_1'(0) = A - B = 0 \Rightarrow y_1 = e^t\cos t - e^t\sin t;\ y_2(0) = A = 0,\ y_2'(0) = A = 1 \Rightarrow y_2 = e^t\cos t.$$

(b) The particular solution will be of the form,

$$y_p = (c_1 + c_2 t)e^t + e^{2t}(c_3\cos 2t + c_4\sin 2t).$$

(6) We first convert this into standard form,

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = -2(t-1)e^{-t}.$$

The Wronskian gives,

$$W(y_1, y_2) = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = e^t - te^t.$$

And recall our equation,

$$y = -y_1\int \frac{y_2 f(t)}{W(y_1, y_2)}dt + y_2\int \frac{y_1 f(t)}{W(y_1, y_2)}dt.$$

(a) We do part b first because we're smart and know how to do things better than the book.

$$-\int \frac{y_2 f(t)}{W(y_1, y_2)}dt = -\int \frac{t[\cancel{-}2\cancel{(t-1)}e^{-t}]}{\cancel{(1-t)}e^t}dt = -2\int te^{-2t}dt = \frac{1}{2}e^{-2t}(2t+1) + C_1,$$

$$\int \frac{y_1 f(t)}{W(y_1, y_2)}dt = \int \frac{\cancel{e^t}[\cancel{-}2\cancel{(t-1)}e^{-t}]}{\cancel{(1-t)}e^t}dt = 2\int e^{-t}dt = -2e^{-t} + C_2.$$

Then our solution is,

$$y = \frac{1}{2}e^{-t}(2t+1) + C_1 e^t - 2te^{-t} + C_2 t = -te^{-t} + C_3 e^t + C_2 t.$$

(b) Then the particular solution is, $y_p = -te^{-t}$.