FALL 2007 SOLUTIONS

(1) Won't be on this exam, was on previous exam.

(2) (a) Using the definition,

$$\int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{t(1-s)}dt + 2\int_1^\infty e^{-st}dt = \frac{1}{1-s}e^{t(1-s)}\Big|_0^1 + 2\lim_{\tau\to\infty}\int_1^\tau e^{-st}dt$$
$$= \frac{1}{1-s}e^{1-s} - \frac{1}{1-s} + 2\lim_{\tau\to\infty} -\frac{1}{s}e^{-st}\Big|_1^\tau = \frac{1}{1-s}\left(e^{1-s}-1\right) + 2\lim_{t\to\infty} -\frac{1}{s}e^{-s\tau} + \frac{1}{s}e^{-s}$$
$$= \frac{1}{1-s}\left(e^{1-s}-1\right) + \frac{2}{s}e^{-s}.$$

(b) Notice $\mathcal{L}\{e^{-t}\} = 1/(s+1)$ and $\mathcal{L}\{\sin t\} = 1/(s^2+1)$, then

$$\mathcal{L}\{g(t)\} = \frac{1}{(s+1)(s^2+1)}.$$

(3) (a) There are two ways you can do this. You can either use a convolution,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}, \mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2-4}\right\} = \frac{1}{2}\sinh 2t \Rightarrow f(t) = \int_0^t \frac{1}{2}\sinh(2\tau)e^{\tau-t}d\tau.$$

Or you can use partial fractions,

$$\frac{1}{(s+1)(s^2-4)} = \frac{1}{(s+1)(s-2)(s+2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}$$
$$\Rightarrow A(s-2)(s+2) + B(s+1)(s+2) + C(s+1)(s-2)$$
$$= A(s^2-4) + B(s^2+3s+2) + C(s^2-s-2)$$
$$= (A+B+C)s^2 + (3B-C)s - 4A + 2B - 2C = 1$$
$$\Rightarrow C = 3B \Rightarrow A = -4B \Rightarrow 16B + 2B - 16B = 12B = 1 \Rightarrow B = 1/12 \Rightarrow C = 1/4,\ A = -1/3.$$

then,

$$f(t) = -\frac{1}{3}e^{-t} + \frac{1}{12}e^{2t} + \frac{1}{4}e^{-2t}.$$

(b) Here we need to manipulate the expression,

$$\frac{s+3}{s^2+4s+8}e^{-4s} = \frac{s+3}{(s+2)^2+4}e^{-4s} = \left[\frac{s+2}{(s+2)^2+4} + \frac{1}{2}\cdot\frac{2}{(s+2)^2+4}\right]e^{-4s}$$
$$\Rightarrow g(t) = \left[e^{-2(t-4)}\cos(2(t-4)) + \frac{1}{2}e^{-2(t-4)}\sin(2(t-4))\right]u_4(t).$$

(4) (a) We take the Laplace transform of the entire ODE,

$$-\cancelto{0}{y'(0)} - s\cancelto{1}{y(0)} + s^2Y - Y = e^{-s} \Rightarrow (s^2-1)Y = e^{-s} + s$$
$$\Rightarrow Y = \frac{e^{-s}}{s^2-1} + \frac{s}{s^2-1} \Rightarrow y = \sinh(t-1)u_1(t) + \cosh t.$$

(b) Again,

$$(s^2+1)Y = \frac{2}{s}e^{-s} - \frac{4}{s}e^{-3s} \Rightarrow Y = \frac{2e^{-s}}{s(s^2+1)} - \frac{4e^{-3s}}{s(s^2+1)}.$$

Notice,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

then

$$Y = \left[\frac{1}{s} - \frac{s}{s^2+1}\right]\left(2e^{-s} - 4e^{-3s}\right) \Rightarrow y = 2u_1(t) - 2\cos(t-1)u_1(t) - 4u_3(t) + 4\cos(t-3)u_3(t).$$

(5) (a) Again,

$$(s^2+4)Y = G(s) \Rightarrow Y = \frac{G(s)}{s^2+4} = \frac{1}{2}\cdot\frac{2}{s^2+4}G(s) \Rightarrow y = \int_0^t \frac{1}{2}\sin(2\tau)g(t-\tau)d\tau.$$

(b) This we skip.

(6) We skip this too.

Spring 2012 Solutions

(1) Recall,

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \Rightarrow y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

(a) Plugging into the ODE gives,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2(n+1)a_{n+1}x^{n+1} + 2a_n x^n = 0.$$

Then we get the following recurrence relations,

$$x^0: \qquad 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$$

$$x^m: \qquad (m+2)(m+1)a_{m+2} + 2ma_m + 2a_m = 0 \Rightarrow a_{m+2} = \frac{-2\cancel{(m+1)}a_m}{(m+2)\cancel{(m+1)}} = -2\frac{a_m}{m+2}.$$

(b) The first few terms of the solutions are,

$$a_0 = 0 \Rightarrow a_2 = a_4 = \cdots = 0,\ a_3 = -\frac{2}{3}a_1 \Rightarrow a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1$$

$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0,\ a_4 = -\frac{1}{2}a_2 = \frac{1}{2}a_0.$$

$$\Rightarrow y_1 = a_1 x - \frac{2}{3}a_1 x^3 + \frac{4}{15}a_1 x^5 + \cdots,\ y_2 = a_0 - a_0 x^2 + \frac{1}{2}a_0 x^4 + \cdots$$

(2) (a) The characteristic polynomial is,

$$6r(r-1) + 7r - 2 = 6r^2 - 6r + 7r - 2 = 6r^2 + r - 2 = 0 \Rightarrow r = \frac{1}{2}, -\frac{2}{3}.$$

Then our solution is,

$$y = c_1|x|^{-1/2} + c_2 x^{-2/3}.$$

(b) We first put the equation in standard form,

$$y'' - \frac{x-4}{x(x+3)}y' - \frac{5}{x(x+3)}y = 0 \Rightarrow x_0 = 0, -3;\ P(x) = -\frac{x-4}{x(x+3)},\ Q(x) = -\frac{5}{x(x+3)}.$$

For $x_0 = 0$ we have,

$$\lim_{x\to 0} xP(x) = \lim_{x\to 0}\frac{4-x}{x+3} = \frac{4}{3};\ \lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0}\frac{-5x}{x+3} = 0.$$

So, $x_0 = 0$ is a regular singular point. For $x_0 = -3$ we have,

$$\lim_{x\to -3}(x+3)P(x) = \lim_{x\to -3}\frac{4-x}{x} = -\frac{7}{3};\ \lim_{x\to -3}(x+3)^2 Q(x) = \lim_{x\to -3}\frac{-5(x+3)}{x} = 0.$$

So, $x_0 = -3$ is a regular singular point.

(3) (a) We need to manipulate the expression first,

$$\frac{s-2}{s^2+6s+25} = \frac{s-2}{(s+3)^2+16} = \frac{s+3}{(s+3)^2+16} - \frac{5}{4}\cdot\frac{4}{(s+3)^2+16}$$

Then the inverse Laplace Transform is,

$$f(t) = e^{-3t}\cos 4t - \frac{5}{4}e^{-3t}\sin 4t.$$

(b) We take the Laplace Transform of the entire ODE,

$$-\cancelto{2}{y'(0)} - s\cancelto{-1}{y(0)} + s^2Y - 4\cancelto{-1}{y(0)} + 4sY + 4Y = 0 \Rightarrow (s+2)^2Y = -2-s \Rightarrow Y = \frac{-2-s}{(s+2)^2} = -\frac{\cancel{(s+2)}}{(s+2)^{\cancel{2}}}$$

Then our solution is,

$$y = -e^{-2t}.$$

(4) Again taking the Laplace Transform,

$$(s^2+3s+2)Y = \frac{1}{s} + \frac{e^{-2s}}{s} - 2\frac{e^{-3s}}{s} \Rightarrow Y = \frac{1+e^{-2s}-2e^{-3s}}{s(s+2)(s+1)}.$$

We apply partial fractions,

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} \Rightarrow A(s+2)(s+1) + Bs(s+1) + Cs(s+2) = 1$$
$$\Rightarrow A(s^2+3s+2) + B(s^2+s) + C(s^2+2s) = (A+B+C)s^2 + (3A+B+2C)s + 2A = 1.$$
$$\Rightarrow A = \frac{1}{2}.B = -A-C \Rightarrow 2A+C = 0 \Rightarrow C = -1 \Rightarrow B = \frac{1}{2}.$$

Then,

$$Y = \left[\frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}\right]\left(e^{-2s} - 2e^{-3s} + 1\right).$$

Then taking the inverse Laplace Transform gives,

$$y = \frac{1}{2} - e^{-t} + e^{-2t} + \left[\frac{1}{2} + \frac{1}{2}e^{-2(t-2)} - e^{-(t-2)}\right]u_2(t) - 2\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-3)} - e^{-(t-3)}\right]u_3(t).$$

(5) We first convert $g(t)$ into step functions, $g(t) = 1 - u_4(t)$, then we take the Laplace Transform,

$$(s^2 - 3s + 2)Y = \frac{1}{s} - \frac{e^{-4s}}{s} \Rightarrow Y = \frac{1 - e^{-4s}}{s(s-1)(s-2)}.$$

We employ partial fractions,

$$\frac{1}{s(s-2)(s+-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-1} \Rightarrow A(s-2)(s-1) + Bs(s-1) + Cs(s-2) = 1$$

$$\Rightarrow A(s^2 - 3s + 2) + B(s^2 - s) + C(s^2 - 2s) = (A + B + C)s^2 - (3A + B + 2C)s + 2A = 1.$$

$$\Rightarrow A = \frac{1}{2}.B = -A - C \Rightarrow 2A + C = 0 \Rightarrow C = -1 \Rightarrow B = \frac{1}{2}.$$

Then we get,

$$Y = \left[\frac{1/2}{s} + \frac{1/2}{s-2} - \frac{1}{s-1}\right]\left(1 - e^{-4s}\right) \Rightarrow y = \frac{1}{2} + \frac{1}{2}e^{2t} - e^t - \left[\frac{1}{2} + \frac{1}{2}e^{2(t-4)} - \frac{1}{2}e^{t-4}\right]u_4(t).$$

(6) (a) We take the Laplace Transform,

$$-\cancelto{1}{y'(0)} - s\cancelto{1}{y(0)} + s^2Y - Y = e^{-5s} \Rightarrow Y = \frac{1 + s + e^{-5s}}{s^2 - 1} = \frac{1}{s-1} + \frac{e^{-5s}}{s^2 - 1}.$$

Then our solution is,

$$y = \sinh t + \cosh t + \sinh(t-5)u_5(t) = e^t + \sinh(t-5)u_5(t).$$

(b) By the definition of Laplace Transforms,

$$\int_0^\infty e^{-st} f(t)dt = \int_0^1 te^{-st}dt + \int_1^2 e^{-st}dt = -\frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^2$$

$$= \left(\frac{1}{s^2} - \frac{1}{s}\right)e^{-s} - \frac{1}{s^2} - \frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s} = \frac{1}{s^2}e^{-s} - \frac{1}{s^2} - \frac{1}{s}e^{-2s}.$$