Fall 2007 Solutions

(1) (a) (i) 2nd order because of y'', nonlinear because of $y'y$.
(ii) 1st order because of y', nonlinear because $1/y$ is nonlinear (think of it's Taylor series); you could also just multiply out by y and get yy' which is obviously nonlinear.

(b) As per usual the characteristic polynomial $r^2 + r - 6 = (r+3)(r-2)$, so our general solution is $y = c_1 e^{-3x} + c_2 e^{2x}$. Plugging in the initial conditions gives $y(0) = c_1 + c_2 = 1$ and $y'(0) = -3c_1 + 2c_2 = k$, solving the system of simultaneous equations gives $c_2 = (k+3)/5$ and $c_1 = (2-k)/5$. This gives a solution of,

$$y = \frac{2-k}{5}e^{-3x} + \frac{k+3}{5}e^{2x}.$$

Hence, notice if $k = -3$, the positive exponential term disappears, so $y \to 0$ as $t \to \infty$ if $k = -3$.

(2) (a) This is clearly separable, so we separate and integrate,

$$\int \frac{e^y dy}{e^y + 1} = \int \frac{dt}{t}$$

We find the first integral by u-sub, with $u = e^y + 1 \Rightarrow du = e^y dy$,

$$\int \frac{e^y dy}{e^y + 1} = \int \frac{du}{u} = \ln u = \ln(e^y + 1).$$

This gives,

$$\ln(e^y + 1) = \ln t + C \Rightarrow e^y + 1 = kt \Rightarrow y = \ln(kt - 1).$$

(b) We will do this problem the same way we derived the characteristic polynomial. Plugging in $y = (t-1)^r$ and $y'' = r(r-1)(t-1)^{r-2}$ gives,

$$(t-1)^2\left[r(r-1)(t-1)^{r-2}\right] - 2(t-1)^r = (t-1)^r\left(r^2 - r - 2\right) = 0 \Rightarrow r^2 - r - 2 = 0 \Rightarrow r = -1, 2.$$

(3) (a) Here we plug in $y = 1/2$ and $y' = 0$,

$$\frac{a}{2} + b = 0 \Rightarrow b = -\frac{a}{2} \Rightarrow y' = -\left(ay - \frac{a}{2}\right)$$

You can pick $a = 2 \Rightarrow b = -1$.

(b) The "r" values are $r = 9, -2$, so $(r-9)(r+2) = r^2 - 7r - 18 \Rightarrow y'' - 7y' - 18y = 0$.

(4) (a) skip - on the next exam

(b) This is in integrating factor form and it's clearly not separable, so we use method of integrating factors,

$$\mu = \exp\left(\int^t \frac{d\tau}{\tau}\right) = \exp(\ln t) = t.$$

using this to convert our ODE we get,

$$d(ty) = t\cos t dt \Rightarrow \int d(ty) = \int t\cos t dt.$$

We integrate the RHS via by-parts, with $u = t \Rightarrow du = dt$ and $dv = \cos t \Rightarrow v = \sin t$,

$$ty = t\sin t - \int \sin t dt = t\sin t + \cos t + C \Rightarrow y = \sin t + \frac{1}{t}\cos t + \frac{C}{t}.$$

(5) (a) This is separable, so we separate and integrate,

$$\int e^{-2y} dy = \int dt \Rightarrow -\frac{1}{2}e^{-2y} = t + C.$$

The initial condition gives $C = -1/2$, so

$$e^{-2y} = 1 - 2t \Rightarrow -2y = \ln(1-2t) \Rightarrow y = -\frac{1}{2}\ln(1-2t).$$

(b) $t \in (-\infty, 1/2)$.

(6) (a) This is separable, so we separate and integrate,

$$\int \frac{2dy}{y^2 - 4} = \int dt.$$

To solve the first integral we have to use partial fractions, but this is easy enough that we can do the partial fractions in our head,

$$\frac{2}{y^2-4} = \frac{2}{(y-2)(y+2)} = \frac{1/2}{y-2} - \frac{1/2}{y+2},$$

which gives,

$$\int \frac{dy}{y-2} - \int \frac{dy}{y+2} = 2\int dt \Rightarrow \ln(y-2) - \ln(y+2) = 2t + C$$
$$\Rightarrow \ln\left(\frac{y-2}{y+2}\right) = 2t + C \Rightarrow \frac{y-2}{y+2} = ke^{2t}.$$

from the initial condition we get $k = -1$, so our solution is

$$\frac{y-2}{y+2} = -e^{2t}.$$

(b) While we can't solve this explicitly for y we can extract information from the form we have. Notice that if we take $t \to 0$ we get $\frac{y-2}{y+2} \to \infty$, so this doesn't help us, but we can take the reciprocal, which will make the problem easier,

$$\lim_{t\to\infty} \frac{y+2}{y-2} = \lim_{t\to\infty} -e^{-2t} = 0.$$

Hence, as $t \to 0$, $y \to -2^+$. It approaches from the right because the RHS is negative. I'll explain this more in class.

(1) (a) skip - next exam
    (b) (i) 3rd order because of y''', nonlinear because of $\sin(x+y)$, again think of the Taylor series of sine.
        (ii) 2nd order because of $\frac{d}{dt}\left(t\frac{dy}{dt}\right)$, and linear.

(2) It might be best if I explain this in class, so here I'm not going to do too much explaining, I'll just give the results. From proportionality we get $dx/dt = rx$ where r is the proportionality constant. We solve this via separation to get,

$$\frac{dx}{dt} = rx \Rightarrow \int \frac{dx}{x} = r\int dt \Rightarrow \ln x = rt + C.$$

Notice we have to constants to solve for: the constant of integration and the proportionality constant. For $t = 3$ we have $\ln(400) = 3r + C$ and for $t = 10$ we have $\ln(2000) = 10r + C$. This gives us two simultaneous equations to solve, which gives $r = (\ln 5)/7$ and $C = \ln(400) - 3(\ln 5)/7$. Plugging this back in gives,

$$\ln x = \frac{t}{7}\ln 5 + \ln(400) - \ln 5^{3/7} \Rightarrow \ln x = \ln 5^{t/7} + \ln(400) - \ln 5^{3/7} = \ln\left(400 \cdot 5^{(t-3)/7}\right)$$
$$\Rightarrow x = (400)5^{(t-3)/7} = 400e^{(\ln 5)(t-3)/7}.$$

(3) (a) We plug in for y and y' as per usual,

$$3a + b = 0 \Rightarrow b = -3a \Rightarrow y' = -(ay - 3a).$$

You can pick $a = 1 \Rightarrow b = -3$.

(b) We need to put this into standard form, $y' - y/x = 1$, this is clearly not separable and it's in the integrating factor form, so we find the integrating factor,

$$\mu = \exp\left(-\int^x \frac{d\xi}{\xi}\right) = e^{-\ln x} = \frac{1}{x}.$$

Now, we proceed with the method of integrating factors,

$$\int d\left(\frac{y}{x}\right) = \int \frac{dx}{x} \Rightarrow \frac{y}{x} = \ln|x| + C \Rightarrow y = x\ln|x| + Cx.$$

From the initial condition we get, $C = 1$, so $y = x\ln|x| + x$.

(4) As per usual we find our characteristic polynomial: $r^2 - 2r - 3 = (r-3)(r+1)$, so $y = c_1e^{3x} + c_2e^{-x}$. From the initial conditions we get, $y(0) = c_1 + c_2 = \alpha$ and $y'(0) = 3c_1 - c_2 = 1$, so we have $4c_1 = \alpha + 1$. We don't even have to solve for the other constant, we have our answer already. Notice that if $\alpha = -1$, the first term disappears and hence $y \to 0$ as $t \to \infty$.

(5) This is a bit harder than the concentration problems we did, but it's a really fun problem.

(a) Solving for the amount of time it takes to fill up is easy because it's only arithmetic. Notice that the volume increases as a constant every minute, i.e. 4 gal. is coming in and 3 gal. is leaving. And we know that 30 gal. of solution needs to enter the tank to fill it to the top, so $T = (30 \text{ gal.}) \div (4 \text{ gal/min} - 3 \text{ gal/min}) = 30$ min.

(b) This is a bit harder, but rate in is easy to find: (2 lb/gal) $\times$(4 gal/min) $=$ 8 lb/min. To find the rate out will take a bit more effort. Notice that in most of our problems the total volume was constant, however here the volume increases as time. Notice that every minute the volume increases by a gallon, so at any time t the volume will be $90 + t$. Now that we have that we move forward as usual: the concentration is $x/(90+t)$ lb/gal, so we get $(x/(90+t)$ lb/gal$)$ $\times$ (3) gal/min $= \frac{3x}{90+t}$ lb/min. Then, our IVP becomes,

$$\frac{dx}{dt} = 8 - \frac{3x}{90+t} \Rightarrow \frac{dx}{dt} + \frac{3x}{90+t} = 8; \; x(0) = 90.$$

We use integrating factors to solve this. The integrating factor is,

$$\mu = \exp\left(\int^t \frac{3d\tau}{90+\tau}\right) = \exp\left(3\ln(90+t)\right) = (90+t)^3.$$

Applying the method of integrating factors gives,

$$\int d\left[(90+t)^3 x\right] = \int 8(90+t)^3 dt \Rightarrow (90+t)^3 x = 2(90+t)^4 + C \Rightarrow x = 2(90+t) + \frac{C}{(90+t)^3}.$$

From the initial condition we get, $C = -90^4$, so for any time the amount of salt is,

$$x = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

(6) (a) We get $y_1' = -3t^{-4}$ and $y_1'' = 12t^{-5}$, and plugging this back in gives, $12t^{-3} - 9t^{-3} - 3t^{-3} = 0$, but notice this is only valid for $t > 0$.

(b) skip - on next exam

(c) skip - on next exam