FALL 2014 SOLUTIONS

(1) (a) 2nd order because $y''$, nonlinear because ${y'}^3$.
(b) 2nd order because $y''$, linear.

(2) (a) $2a+b=0 \Rightarrow b=-2a \Rightarrow y' = ()(ay-2a)$. Since it's converging, $y' = -(ay-2a)$, so pick $a=1 \Rightarrow y' = -y+2$.
(b) We first convert this to standard form, $y' + 2y/t = t - 1 + 1/t$. We employ the method of integrating factors,

$$\mu = \exp\left(2\int^t \frac{d\tau}{\tau}\right) = e^{2\ln t} = t^2.$$

We put this back into the form of integrating factors,

$$\int d(t^2y) = \int (t^3 - t^2 + t)dt \Rightarrow t^2y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C \Rightarrow y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}.$$

From the initial condition we get $y(1) = 1/4 - 1/3 + 1/2 + C = 1/2 \Rightarrow C = 1/12$. This gives us a solution of,

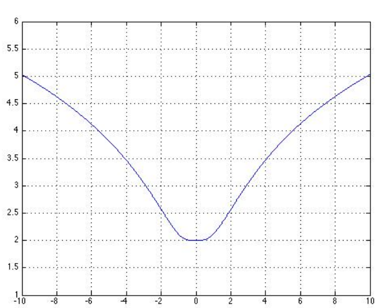
$$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}.$$

(3) (a) This is clearly separable so,

$$\int ydy = \int \frac{2x}{1+x^2}dx \Rightarrow \frac{1}{2}y^2 = \ln(1+x^2) + C_0 \Rightarrow y^2 = \ln(1+x^2)^2 + C_1.$$

From the initial condition we readily get $C_1 = 4$. We also get that we must choose the positive branch of the square root, so we get

$$y = \sqrt{\ln(1+x^2)^2 + 4}.$$



(b) We must first put this into standard form: $y' + \frac{t+1}{t}y = 1$. Then we get an integrating factor of,

$$\mu = \exp\left(\int^t (1 + \frac{1}{\tau})d\tau\right) = e^{t+\ln t} = te^t.$$

Putting this into the integrating factor equation gives,

$$\int d(te^t y) = \int te^t dt \Rightarrow te^t y = te^t - e^t + C.$$

From the initial condition we have $2\ln 2 - 2 + C = 2\ln 2 \Rightarrow C = 2$. This gives,

$$y = 1 - \frac{1}{t} + \frac{2}{t}e^{-t}.$$

Finally, $y \to 1$ as $t \to \infty$.

(4) Told ya we'd have a concentration problem!

(a) (8 L) $\div$ 2 L/Hr = 4 Hr.

(b) Rate in is 6 g/Hr and the rate out is $x/(1+t)$ g/Hr, then

$$\frac{dx}{dt} + \frac{x}{1+t} = 6;\ x(0) = .2$$

We solve this via integrating factors,

$$\mu = \exp\left(\int^t \frac{d\tau}{1+\tau}\right) = e^{\ln(1+t)} = 1 + t.$$

Then,

$$\int d((1+t)x) = \int (6+6t)dt \Rightarrow (1+t)x = 6t + 3t^2 + C = 3t(2+t) + C.$$

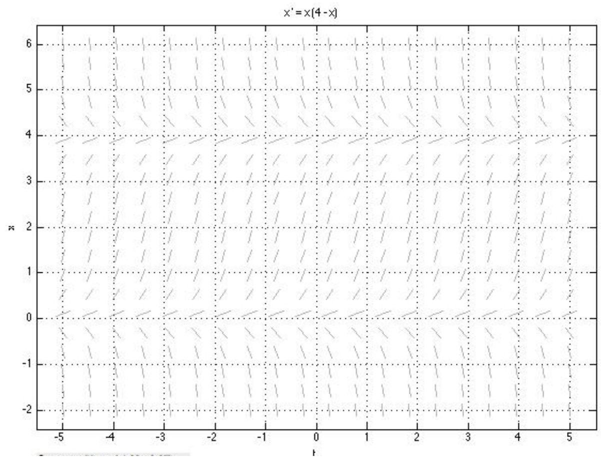
The initial conditions readily give, $C = .2$, so our solution is

$$x = 3t\frac{2+t}{1+t} + \frac{1}{5(1+t)}.$$

Then the final amount of salt is $x(4) = 12\cdot(6/5) + 1/25 = 361/25$ g. Finally, the concentration is $x(4)/10 = 361/250$ g/L.

(5) (a) 1st order, nonlinear

(b) We first find the fixed points $y_*(4 - y_*) = 0 \Rightarrow y_* = 0, 4$. Then we find the sign of $dy/dt$ on respective intervals, $dy/dt < 0$ on $(-\infty, 0)$ and $(4, \infty)$ and $dy/dt > 0$ for $(0, 4)$. As $t \to \infty$, $y \to -\infty$ for $y < 0$, and $y \to 4$ for $0 < y < 4$ and $y > 4$. This is more clearly seen in the direction field sketch,




(c) To solve we simply separate and integrate,

$$\int \frac{dy}{y(4-y)} = \frac{1}{4}\int \left(\frac{1}{y} + \frac{1}{4-y}\right) = \int dt = t + C \Rightarrow \frac{1}{4}(\ln y - \ln(4-y)) = t + C_0 \Rightarrow \ln \frac{y}{4-y} = 4t + C_1.$$

From the initial condition we readily get $C_1 = \ln a/(4-a)$, then

$$\frac{y}{4-y} = \frac{a}{4-a}e^{4t}.$$

This doesn't allow us to make a conclusion on the behavior of $y$ as $t \to \infty$ readily, but we can take the reciprocal and see what happens to that,

$$\frac{4-y}{y} = \frac{4-a}{a}e^{-4t},$$

so for $a \in (0, 4)$: $y \to 4^-$ and for $a \in (4, \infty)$: $y \to 4^+$. Of course if $a = 4$, $y = 4$ for all time.

(6) Oh Thank God, I can finally get some sleep.

(a) We get the characteristic polynomial $(r-2)(r+3) = r^2 + r - 6 \Rightarrow y'' + y' - 6y = 0$.

(b) We go straight to the characteristic polynomial: $r^2 - r - 2 = (r-2)(r+1) = 0$, then $y = c_1 e^{2x} + c_2 e^{-x}$. The initial conditions give us $y(0) = c_1 + c_2 = \alpha$ and $y'(0) = 2c_1 - c_2 = 2$, which gives $3c_1 = \alpha + 2$, then the only way to get rid of the part that blows up is to make $c_1 = 0 \Rightarrow \alpha = -2$. Notice, that we don't have to solve the ODE.