(1) (a) We take the Wronskian,

$$W(f,g) = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g = 3e^{4t}.$$

Then we solve the ODE,

$$g' - 2g = 3e^{2t} \Rightarrow \mu = \exp\left(-\int^t 2d\tau\right) = e^{-2t} \Rightarrow e^{-2t}g = \int^t 3d\tau = 3t \Rightarrow g = 3te^{2t}.$$

(b) (i) From the Wronskian we get that they are L.I.,

$$W(y_1,y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0.$$

(ii) From the Wronskian we get that they are L.I.,

$$W(y_1,y_2) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x^2e^x \neq 0.$$

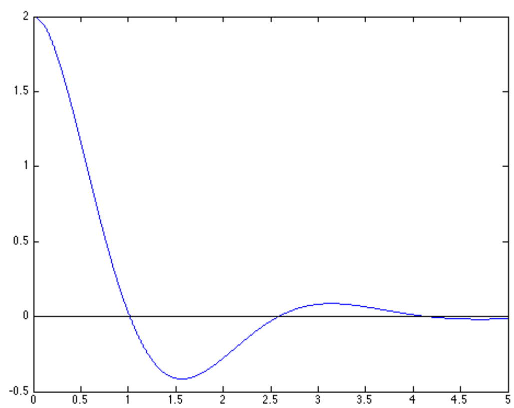
(2) The characteristic polynomial gives, $r^2 + 2r + 5 = 0 \Rightarrow r = -1 \pm 2i$, then our general solution is

$$y = e^{-t}(A\cos 2t + B\sin 2t)$$

Our initial conditions give us, $y(0) = A = 2$ and $y'(0) = -2 + 2B = 0 \Rightarrow B = 1$, then our solution is

$$y = e^{-t}(2\cos 2t + \sin 2t).$$

Your sketch should look like,



(3) Plugging the first solution into the ODE gives $t^2(2t^{-3}) + 3t(-t^{-2}) + t^{-1} = 0$. Now, for reduction of order let $y = v(t)t^{-1}$, which gives the derivatives,

$$y' = v't^{-1} - t^{-2}v \Rightarrow y'' = v''t^{-1} - 2t^{-2}v' + 2t^{-3}.$$

Plugging in to the ODE gives,

$$v''t - 2v' + 2t^{-1} + 3v' - 3vt^{-1} + vt^{-1} = tv'' + v' = 0.$$

Let $u = v'$, then we get,

$$u' = -\frac{1}{t}u \Rightarrow \ln u = \int \frac{dt}{t} = -\ln t + C_0 \Rightarrow u = \frac{k}{t} \Rightarrow v = k\ln t + C_1$$

Then, $y = k\ln t/t + C_1/t$, so $y_2 = \ln t/t$.

(4) (a) We employ the characteristic polynomial, $2r^2 + 3r + 1 = 0 \Rightarrow r = -3/4 \pm 1/4 = -1, -1/2$. Then the characteristic solution is, $y_c = c_1e^{-t/2} + c_2e^{-t}$. Then our particular solution is,

$$y_p = A_1t^2 + A_2t + A_3 + B_1\cos t + B_2\sin t \Rightarrow y_p' = 2A_1t + A_2 - B_1\sin t + B_2\cos t \Rightarrow y_p'' = 2A_1 - B_1\cos t - B_2\sin t.$$

Plugging into the ODE gives,

$$4A_1 - 2B_1\cos t - 2B_2\sin t + 6A_1t + 3A_2 - 3B_1\sin t + 3B_2\cos t + A_1t^2 + A_2t + A_3 + B_1\cos t + B_2\sin t = t^2 + 3\sin t.$$

Grouping terms gives,

$$A_1t^2 + (A_2 + 6A_1)t + (4A_1 + 3A_2 + A_3) + (3B_2 - B_1)\cos t - (3B_1 + B_2)\sin t = t^2 + 3\sin t.$$

Then we get $A_1 = 1$, $A_2 = -6$, $A_3 = 14$, $B_1 = -9/10$, $B_2 = -3/10$. Then our full general solution is,

$$y = c_1e^{-t/2} + c_2e^{-t} + t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t.$$

(b) We have to solve for the characteristic solution first, $r^2 + 3r = r(r+3) = 0$, then $y_c = c_1 + c_2e^{-3t}$, so our particular solution is,

$$y_p = t(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0) + t(B_2t^2 + B_1t + B_0)e^{-3t} + D_1\sin 3t + D_2\cos 3t.$$

(5) Plugging $y_1 = t$ into the ODE gives, $-t(t+2) + t(t+2) = 0$ and $y_2 = te^t$ gives,

$$2t^2e^t + t^3e^t - t^2e^t - t^3e^t - 2te^t - 2t^2e^t + t^2e^t + 2te^t = 0.$$

Taking the Wronskian gives,

$$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t.$$

We recall our variation of parameters formula,

$$y = -y_1\int \frac{y_2f(t)}{W(y_1,y_2)}dt + y_2\int \frac{y_1f(t)}{W(y_1,y_2)}dt.$$

We do the integrals individually,

$$\int \frac{y_2f(t)}{W(y_1,y_2)}dt = \int \frac{(te^t)(2t)}{t^2e^t}dt = -2t + C_1; \qquad \int \frac{y_1f(t)}{W(y_1,y_2)}dt = \int \frac{t(2t)}{t^2e^t}dt = 2\int e^{-t}dt = -2e^{-t} + C_2$$

Plugging these back into our equation gives,

$$y = -2t^2 + C_1t - 2t + C_2te^t = -2t^2 + C_3t + C_2te^t.$$

Which means, $y_p = -2t^2$.

(6) (a) Our IVP is,

$$\frac{1}{2}u'' + 2u' + 64u = 0;\ u(0) = 0,\ u'(0) = \frac{1}{4}.$$

(b) Our characteristic polynomial gives, $r^2/2 + 2r + 64 = 0 \Rightarrow r = -2 \pm i\sqrt{124}$. Then our general solution is,

$$u = e^{-2t}(A\cos\sqrt{124}t + B\sin\sqrt{124}t).$$

Then from the initial conditions we get, $u(0) = A = 0$ and $u'(0) = B\sqrt{124} = 1/4$, so $B = 1/4\sqrt{124}$. Then our solution is,

$$u = \frac{1}{4\sqrt{124}}e^{-2t}\sin\sqrt{124}t$$

(c) Your plot should look as follows,