(1) We set up the problem and plug into the ODE,

$$y=\sum_{n=0}^{\infty}a_nx^n \Rightarrow y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \Rightarrow y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}4(n+2)(n+1)a_{n+2}x^n-(n+2)(n+1)a_{n+2}x^{n+2}+2a_nx^n=0.$$

(a) For this case the general recurrence suffices,

$$4(m+2)(m+1)a_{m+2}-m(m-1)a_m+2a_m=0 \Rightarrow a_{m+2}=\frac{m^2-m-2}{4(m+2)(m+1)}a_m=\frac{(m-2)\cancel{(m+1)}}{4(m+2)\cancel{(m+1)}}a_m$$

(b) Now for the solutions, when $a_0=0$, all the even indices are zero, so $a_3=-a_1/12$ and $a_5=a_3/20=-a_1/240$. If $a_1=0$, all the odd indices are zero, so $a_2=-a_0/4$, but it terminates here since if we plug in $m=2$ to get a_4 we see that $a_4=0$, so all other even terms are zero. So we get,

$$y_1=x-\frac{1}{12}x^3-\frac{1}{240}x^5+\cdots;\ y_2=1-\frac{1}{4}x^2$$

Notice, it doesn't matter whether or not you include the constants a_0 or a_1.

(2) (a) Our characteristic polynomial gives,

$$2r(r-1)+r-3=2r^2-r-3=0 \Rightarrow r=-1,3/2 \Rightarrow y=\frac{c_1}{x}+c_2|x|^{3/2}.$$

They wont take points off for leaving the absolute values out. From the initial conditions we get, $y(1)=c_1+c_2=1$ and $y'(1)=-c_1+3c_2/2=4$, then $5c_2/2=5 \Rightarrow c_2=2 \Rightarrow c_1=-1$, then we get

$$y=2|x|^{3/2}-\frac{1}{x}.$$

(b) We convert this into standard form,

$$y''+\frac{x-2}{x^2(1-x)}y'-\frac{3\cancel{x}}{x^{\cancel{2}}(1-x)}y=0 \Rightarrow x_0=0,\ 1$$

For $x_0=0$, we have

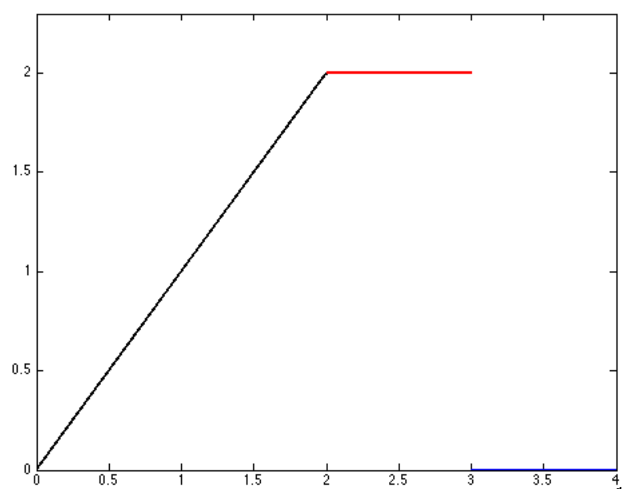
$$\lim_{x\to 0}xP(x)=\lim_{x\to 0}\frac{x-2}{x(1-x)}=\infty \Rightarrow \text{Irregular.}$$

For $x_0=1$,

$$\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{2-x}{x^2}=1\checkmark;\ \lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 1}\frac{3(x-1)}{x}=0\checkmark \Rightarrow \text{Regular}$$

(3) (a) We express it as a step function and take the laplace transform with the plot bellow,

$$f(t)=t(1-u_2(t))+2(u_2(t)-u_3(t))=t-(t-2)u_2(t)-2u_3(t) \Rightarrow F(s)=\frac{1}{s^2}-\frac{e^{-2s}}{s^2}-\frac{2e^{-3s}}{s}.$$



(b) We take the laplace transform,

$$F(s) = \frac{2s+1}{(s-1)^2+1} = 2\frac{s-1}{(s-1)^2+1} + \frac{3}{(s-1)^2+1} \Rightarrow f(t) = 2e^t \cos t + 3e^t \sin t.$$

(4) Taking the laplace transform gives,

$$\left(s^2+4\right) Y = \frac{e^{-\pi s - e^{-2\pi s}}}{s} \Rightarrow Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+4)}.$$

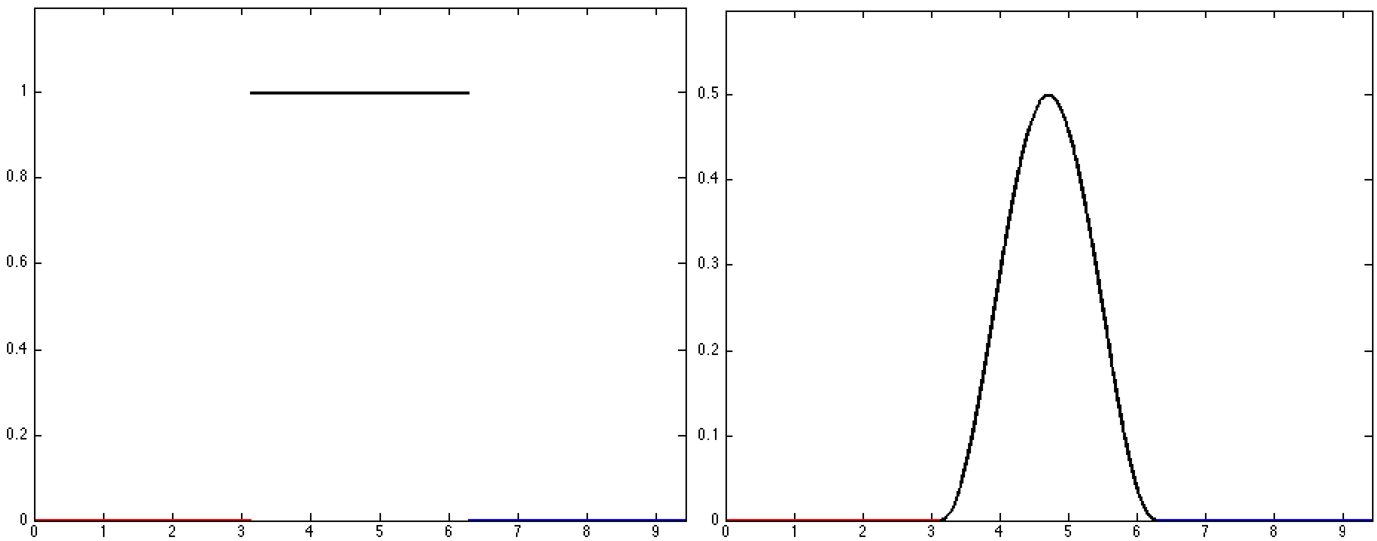
We do the partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \Rightarrow Y = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2-4}\right)\left(e^{-\pi s} - e^{-2\pi s}\right).$$

Taking the inverse laplace transform gives,

$$y = \frac{1}{4}\left[(1-\cos(2(t-\pi)))u_\pi(t) - (1-\cos(2(t-2\pi)))u_{2\pi(t)}\right].$$

The plots of the forcing and the solution are given bellow.



(5) We take the laplace transform,

$$-\cancelto{2}{y'(0)} - s\cancelto{0}{y(0)} + s^2Y - 2\cancelto{0}{y(0)} + 2sY + 5Y = e^{-(\pi/2)s} \Rightarrow -2 + (s^2+2s+5)Y = e^{-(\pi/2)s} \Rightarrow Y = \frac{2+e^{-(\pi/2)s}}{(s+1)^2+4}.$$

Taking the inverse laplace transform gives,

$$y = e^{-t}\sin(2t) + \frac{1}{2}e^{-(t-\pi/2)}\sin\left(2\left(t-\frac{\pi}{2}\right)\right)u_{\pi/2}(t).$$

(6) (a) Notice $\mathcal{L}\{t^2\} = 2/(s^3)$ and $\mathcal{L}\{e^{2t}\} = 1/(s-2)$, then

$$F(s) = \frac{2}{(s-2)s^3}.$$

(b) Notice $\mathcal{L}^{-1}\{1/(s+3)\} = e^{-3t}$ and $\mathcal{L}^{-1}\{s/(s^2+4)\} = \cos(2t)$, then

$$f(t) = \int_0^t e^{-3\tau}\cos(2(t-\tau))d\tau.$$