2.1 Linear Equations; Method of Integration Factor

Consider the ODE $\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0$. This is clearly not separable.

Now consider the ODE $t^2 \frac{dx}{dt} + 2xt = t$. This too is not separable, but we can make it separable by employing a small trick. Notice that $t^2 \frac{dx}{dt} + 2xt = \frac{d}{dt}(xt^2)$, so the ODE becomes, $\frac{d}{dt}(xt^2) = t$, which is separable. This is what is referred to as an "exact ODE". So we get,

$$\frac{d}{dt}(xt^2) = t \Rightarrow d(xt^2) = tdt \Rightarrow \int d(xt^2) = \int tdt \Rightarrow xt^2 = \frac{1}{2}t^2 + C \Rightarrow x = \frac{1}{2} + Ct^{-2}.$$

This is the idea. If we encounter an equation that isn't separable we need to change it in some way that makes it separable.

Lets look at the first equation again and write it in differential form, i.e.

$$\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0 \Rightarrow xdy - ydx + f(x)dx = 0.$$

Notice, that $xdy - ydx$ is almost quotient rule, we just need to finish the denominator, which we notice should be x^2, so let's multiply through by $1/x^2$,

$$\frac{xdy - ydx}{x^2} + \frac{f(x)}{x^2}dx = 0 \Rightarrow d\left(\frac{y}{x}\right) = -\frac{f(x)}{x^2}dx \Rightarrow \int d\left(\frac{y}{x}\right) = -\int \frac{f(x)}{x^2}dx$$
$$\Rightarrow \frac{y}{x} = -\int \frac{f(x)}{x^2}dx \Rightarrow y = -x\int \frac{f(x)}{x^2}dx.$$

This is called the method of "integrating factors", where $1/x^2$ is called the "integrating factor", which are delineated in the following definition.

Definition 1. Consider an ODE of the form

$$\frac{dy}{dx} + p(x)y = g(x). \tag{1}$$

We call $\mu(x)$ an integrating factor if

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right]$$

is an exact ODE, i.e.

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right] \Leftrightarrow d(\mu(x)y) = \mu(x)g(x)dx. \tag{2}$$

All we need to do now is figure out what $\mu(x)$ is in general, but fortunately Leibniz already did that for us,

$$\mu(x) = \exp\left(\int^x p(\xi)d\xi\right). \tag{3}$$

In the following examples we use the method of integrating factors to solve our ODE,

5) The integrating factor is $\mu = \exp\left(\int^t -2ds\right) = e^{-2t}$. Now, we use our method to get,

$$e^{-2t}y = 3\int e^{-t}dt = -3\int e^{-t}dt = -3e^{-t} + C \Rightarrow y = -3e^{t} + Ce^{2t}.$$

Now, notice if $C > 0$, $y \to \infty$ as $t \to \infty$, and if $C \leq 0$, $y \to -\infty$ as $t \to \infty$.

10) The integrating factor is $\mu = \exp\left(\int^t -1/sds\right) = 1/t$, then

$$\frac{y}{t} = \int e^{-t}dt = -e^{-t} + C \Rightarrow y = te^{-t} + Ct.$$

Now, notice if $C = 0$, $y \to 0$ as $t \to \infty$, and if $C \neq 0$, $y \to \infty$ as $t \to \infty$.

21) The integrating factor is $\mu = \exp\left(\int^t -(1/2)ds\right) = e^{-t/2}$. Then,

$$e^{-t/2}y = 2\int e^{-t/2}\cos t dt = \frac{4}{5}e^{-t/2}(2\sin t - \cos t) + C.$$

We did the integration in class. Know how to do the integration! Then, we get

$$y = \frac{4}{5}(2\sin t - \cos t) + Ce^{t/2}.$$

From the initial condition we get $C = a + 4/5$. We see that the behavior of the system changes at $C = 0$, so $a_0 = -4/5$. Now, when $a = -4/5$, y is oscillatory as $t \to 0$, specifically $y \to \frac{4}{5}(2\sin t - \cos t)$. Furthermore, if $a < -4/5$, $y \to -\infty$, and if $a > -4/5$, $y \to \infty$.

2.3 More Modeling Problems

There hasn't been any EE problems yet in the book, so lets do one,

Ex: Consider a Resister-Inductor (RL) circuit in series. Let x be the current at time t. Let V be the voltage across the voltage source, R be the resistance of the resistor, and L be the inductance of the inductor. Now, the voltage drop through the resistor is: $V_R = Rx$, and the voltage through the inductor is $V_L = Ldx/dt$. Now, by Kirchoff's law, we know that the voltages in a loop sum up, so $Ldx/dt + Rx = V$, and in standard form this is,

$$\frac{dx}{dt} + \frac{R}{L}x = \frac{V}{L}.$$

We can solve this via separation,

$$\int \frac{dx}{-Rx/L + V/L} = \int dt \Rightarrow -\frac{L}{R}\ln\left(\frac{V}{L} - \frac{R}{L}x\right) = t + C_0 \Rightarrow \ln\left(\frac{V}{L} - \frac{R}{L}x\right) = -\frac{R}{L}t + C_1$$

$$\Rightarrow \frac{V}{L} - \frac{R}{L}x = \exp\left(-\frac{R}{L}t + C_1\right) = e^{-Rt/L}e^{C_1} = k_0e^{-Rt/L}$$

$$\Rightarrow \frac{R}{L}x = \frac{V}{L} - k_0e^{-Rt/L} \Rightarrow x = \frac{V}{R} - k_1e^{-Rt/L}.$$

Notice that we could use separation because V was constant, however if $V = V(t)$, then we would have to use integrating factor.

The next couple of examples are from the book,

3) Notice that there are two processes delineated in the problem. And the second process starts as soon as the first process ends. So we need to solve the first problem and then use information from the first problem to solve the second problem.

   Process 1: Let x be the amount of salt in lb at time t min. The rate in will be (1/2) lb/gal $\times$ 2 gal/min = 1 lb/min. And the rate out is $x/200$ lb/gal $\times$ 2 gal/min = $x/50$ lb/min. Now notice that there is no salt in the tank when the process starts, so our full IVP becomes,

$$\frac{dx}{dt} = 1 - \frac{x}{50};\ x(0) = 0.$$

   Now we solve this via separation,

$$\int \frac{dx}{1 - x/50} = \int dt \Rightarrow -50\ln(1 - x/50) = t + C_0 \Rightarrow \ln(1 - x/50) = -t/50 + C_1$$
$$\Rightarrow 1 - \frac{x}{50} = k_0 e^{-t/50} \Rightarrow x = 50 - k_1 e^{-t/50}.$$

   Now, we solve for the constant from the initial condition,

$$x(0) = 50 - k_1 - 0 \Rightarrow k_1 = 50 \Rightarrow x = 50\left(1 - e^{-t/50}\right).$$

   Now, since the process is stopped at $t = 10$ min. we need to calculate the amount at that time, $x(10) = 50\left(1 - e^{-1/5}\right)$.

   Process 2: Now, in order to distinguish this process from the previous one, let y be the amount of salt in lb at time t min. Notice, no more salt is entering, so the rate in is zero. The rate out will be the same as before $y/50$ lb/min. For our initial conditions, notice that where this process begins the other one had ended, so $y(0) = x(10)$.

$$\frac{dy}{dt} = -\frac{y}{50};\ y(0) = 50\left(1 - e^{-1/5}\right).$$

   Again, we solve this via separation,

$$\ln y = -\frac{1}{50}t + C \Rightarrow y = ke^{-t/50}.$$

   From the initial condition we have,

$$y(0) = k = 50\left(1 - e^{-1/5}\right) \Rightarrow y = 50\left(1 - e^{-1/5}\right)e^{-t/50}.$$

   Finally, the process stops after another 10 minutes, so $y(10) = 50\left(1 - e^{-1/5}\right)e^{-1/5}$.

8) For this problem we first realize that every year the bank statement increases by k \$ from what the person deposits. However, there is also an interest being earned, which is on the total amount. So, every year the increase due to interest is rS \$. This means the total rate is going to be, $dS/dt = k + rS$. And the initial condition will be $S(0) = 0$.

(a) We solve this via separation,

$$\int \frac{dS}{k+rS} = \int dt \Rightarrow \frac{1}{r}\ln(k+rS) = t + C_0 \Rightarrow \ln(k+rS) = rt + C_1 \Rightarrow k + rS = C_2 e^{rt}.$$

From the initial condition we get,

$$S(0) = 0 \Rightarrow C_2 = k \Rightarrow S = \frac{k}{r}\left(e^{rt} - 1\right).$$

(b) For this problem we solve for $S(40) = 10^6$ with $r = .075$. Plugging all these into the equation gives $k = (.075 \times 10^6)/[\exp(.075 \times 40) - 1]$.

(c) Plug in the values they give and then ask wolfram alpha to solve it.

3.1 Homogeneous Equations with Constant Coefficients

It should be noted that while this chapter is on second order ODEs, we will develop the theory for higher order ODEs because the theory is exactly the same! Let us first go over some definitions we might not know,

Definition 2. An ODE is homogeneous if it is of the form

$$p_n(t)y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_2(t)y''(t) + p_1(t)y'(t) + p_0(t)y(t) = 0. \quad (4)$$

So an example of a second order homogeneous ODE would be
$p_2y'' + p_1y' + p_0y = 0$.

Definition 3. An ODE is said to be nonhomogeneous if it's not homogeneous.

An example of a second order nonhomogeneous ODE would be $p_2y''+p_1y'+p_0y = f(t)$. In this section we will only deal with constant coefficients which mean each $p_n(t) = a_n$ where $a_0, a_1, \ldots, a_{n-1}, a_n$ are all constants.

Now, we consider a special case of Eq. (4): $y' + ay = 0$ We know how to solve this, we simply use separation to get $y = ke^{-ax}$. So, we can "guess" that the form of the solutions for Eq. (4) with constant coefficients will be $y = ke^{rx}$. Now, we plug this guess in to see what the solutions exactly are. Notice that the nth derivative is, $y^{(n)} = kr^ne^{rx}$, so plugging this into (4) with $p_n(t) = a_n$ gives,

$$\begin{aligned} a_nkr^ne^{rx} + a_{n-1}kr^{n-1}e^{rx} + \cdots + a_2kr^2e^{rx} + a_1kre^{rx} + a_0ke^{rx} = 0 \\ \Rightarrow ke^{rx}\left(a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0\right) = 0. \end{aligned}$$

Now, all we have to do is solve the polynomial equation. Since this is an nth order polynomial, there will be n solutions, i.e. $r = r_1, r_2, \ldots, r_{n-1}, r_n$. Since the polynomial equation has n solutions, the ODE will also have n solutions, so by superposition we get,

$$y = k_1e^{r_1x} + k_2e^{r_2x} + \cdots + k_{n-1}e^{r_{n-1}x} + k_ne^{r_nx}.$$

We have just proved a theorem,

Theorem 1. *Consider the ODE*

$$a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0. \qquad (5)$$

such that $a_0, a_1, \ldots, a_{n-1}, a_n$ *are constants. Then,*

$$y = k_1 e^{r_1 x} + k_2 e^{r_2 x} + \cdots + k_{n-1}e^{r_{n-1}x} + k_n e^{r_n x}, \qquad (6)$$

where $k_1, k_2, \ldots, k_{n-1}, k_n$ *are constants and* $r_1, r_2, \ldots, r_{n-1}, r_n$ *satisfy the polynomial equation*

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0, \qquad (7)$$

only if $r_1 \neq r_2 \neq \cdots \neq r_{n-1} \neq r_n$.

Definition 4. We call Eq. (7) the <u>characteristic equation</u> of ODE (5), and the polynomial is called the <u>characteristic polynomial</u>.

Now, lets do a few problems from the book,

1) The characteristic polynomial is $r^2 + 2r - 3$, so

$$r^2 + 2r - 3 = 0 \Rightarrow (r+3)(r-1) = 0 \Rightarrow r = 1, -3 \Rightarrow y = c_2 e^x + c_2 e^{-3x}.$$

7) The characteristic polynomial is $r^2 - 9r + 9$, so

$$r = \frac{1}{2}(9 \pm 3\sqrt{5}) \Rightarrow y = c_1 e^{\frac{1}{2}(9+3\sqrt{5})x} + c_2 e^{\frac{1}{2}(9-3\sqrt{5})x}.$$

12) The characteristic polynomial is $r^2 + 3r$, so

$$r = 0, -3 \Rightarrow y = c_1 + c_2 e^{-3x},$$

and from the initial conditions we get $y = -1 - e^{-3x}$.

18) Here they give us the solution and we have to extract the ODE. Notice that from the solution we deduce

$$r = -\frac{1}{2}, -2 \Rightarrow (r + \frac{1}{2})(r+2) = r^2 + \frac{5}{2}r + 1 = 0 \Rightarrow y'' + \frac{5}{2}y' + y = 0.$$

21) This is kind of a silly question, but since there is a similar one on the homework lets do it. We solve the ODE as per usual,

$$r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r = -1, 2 \Rightarrow y = c_1 e^{-x} + c_2 e^{2x}.$$

From the initial condition we have the equations $c_1 + c_2 = \alpha$ and $2c_2 - c_1 = 2$, so $3c_2 = \alpha + 2$. This means that if $\alpha = -2$, as $t \to \infty$, $y \to 0$. However, for the second part of the problem there are no solutions that always blow up because we have a negative exponential term that will persist.

24) For this problem the ODE itself has the parameter α. This leads to interesting conclusions without even solving, but the easiest most intuitive way to come to those conclusions will be by solving, even though it is more tedious and time consuming. We solve the ODE,

$$r^2 + (3-\alpha)r - 2(\alpha - 1) = 0 \Rightarrow (r - (\alpha-1))(r+2) = 0 \Rightarrow r = -2, \alpha - 1 \Rightarrow y = c_2 e^{-2x} + c_2 e^{(\alpha-1)x}.$$

So, for $\alpha < 1$, $y \to \infty$. If $\alpha = 1$, $y \to c_2$, and if $\alpha > 1$, and $y \to \pm\infty$ only if $c_2 \neq 0$.