3.4 Repeated Roots and Reduction of Order

Repeated Roots: Again consider a second order homogeneous IVP with it's respective characteristic polynomial equation,

$$y'' + by' + cy = 0,\ y(0) = A,\ y'(0) = B; \tag{1}$$
$$r^2 + br + c = 0; \tag{2}$$

Then, our roots (also called eigenvalues) are $r = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. What if $b^2 - 4c = 0$? Then, $r_{1,2} = -b/2$. If we plug this in as usual we get, $y = c_1 e^{-bx/2} + c_2 e^{-bx/2} = (c_1 + c_2)e^{-bx/2}$. However, this only gives us one constant so there is no way we can satisfy the two initial conditions. So, we need another solution in addition to the one we have.

Suppose the "constant" $c_1 + c_2$ is not a constant, but rather a function of x, i.e. $y = v(x)e^{-bx/2}$. We have to figure out if a v will satisfy our ODE, and if so what v is it. We want to plug into (1). The derivatives are,

$$y' = v'(x)e^{-bx/2} - \frac{b}{2}e^{-bx/2}v(x) \Rightarrow y'' = v''(x)e^{-bx/2} - be^{-bx/2}v'(x) + \frac{b^2}{4}e^{-bx/2}v(x).$$

Plugging into the ODE gives,

$$e^{-bx/2}\left(v'' + (-b + b)v' + (\frac{b^2}{4} - \frac{b^2}{2} + \frac{b^2}{4})\right) = e^{-bx/2}v'' = 0.$$

Since $\exp(-bx/2)$ can't be zero, $v'' = 0 \Rightarrow v' = c_3 \Rightarrow v = c_3 x + c_4$, which gives us the solution of,

$$y = (c_3 x + c_4)e^{-bx/2}.$$

This is outlined in the following theorem,

Theorem 1. *Consider the ODE,*

$$ay'' + by' + c = 0. \tag{3}$$

If the characteristic polynomial has repeated roots, i.e. $r_{1,2} = \lambda$, *then the general solution to (3) is,*

$$y = (c_1 + c_2 x)e^{\lambda x}. \tag{4}$$

Proof. Clearly (4) is a solution to (3), which we can easily verify. Furthermore, $W(c_1 e^{\lambda x}, c_2 x e^{\lambda x}) \neq 0$, which we calculated in class. □

Now, lets solve some problems before moving onto the second part of this section.

2) The characteristic equation is $9r^2 + 6r + 1 = 0 \Rightarrow r = -1/3$, then our solution is $y = (c_1 + c_2 x)\exp(-x/3)$.
8) As per usual, $16r^2 + 24r + 9 = 0 \Rightarrow r = -3/4 \Rightarrow y = (c_1 + c_2 x)\exp(-3x/4)$.
12) For this problem we solve the IVP. Our roots are, $r^2 - 6r + 9 = 0 \Rightarrow r = 3$. So, our solution is $y = (c_1 + c_2 x)\exp(3x)$. From the initial conditions we have, $y(0) = c_1 = 0$ and $y'(0) = c_2 = 2$, so our final solution is $y = 2xe^{3x}$, so as $x \to \infty$, $y \to \infty$.
15) Lets only do part d. Solving for the ODE gives, $4r^2 + 12r + 9 = 0 \Rightarrow r = -3/2 \Rightarrow y = (c_1 + c_2 x)\exp(-3x/2)$. The first initial condition gives, $y(0) = c_1 = 1$. The other one gives, $y'(0) = -3/2 + c_2 = b \Rightarrow c_2 = b + 3/2$. So, when $b < -3/2$ it's eventually negative, but when $b > -3/2$ it's always positive.

Reduction of Order: The method used earlier is called *reduction of order*. But, we'll see that this is a far more powerful method than it seems. Consider the ODE,

$$y'' + p(x)y' + q(x)y = 0. \tag{5}$$

Suppose we know one solution of the ODE, call it y_1. Then we "guess" the form of the full solution as $y = v(x)y_1(x)$. First we find the derivatives,

$$y' = y_1'v + v'y_1 \Rightarrow y'' = y_1''v + 2v'y_1' + v''y_1.$$

Plugging this in and grouping the respective v's gives us,

$$\begin{aligned} y_1''v + 2v'y_1' + v''y_1 + py_1'v + pv'y_1 + qy_1v &= y_1v'' + (2y_1' + py_1)v' + \cancelto{0}{(y_1'' + py_1' + qy_1)}v \\ &= y_1v'' + (2y_1' + py_1)v' = 0. \end{aligned}$$

And set $u = v'$. Then we get,

$$\begin{aligned} y_1u' + (2y_1' + py_1)u = 0 &\Rightarrow u' + \frac{2y_1' + py_1}{y_1}u = 0 \Rightarrow \int \frac{du}{u} = -\int \frac{dy_1' + py_1}{y_1}dx \\ &\Rightarrow \ln u = -\int \frac{dy_1' + py_1}{y_1}dx \Rightarrow u = \exp\left(-\int \frac{dy_1' + py_1}{y_1}dx\right) \\ &\Rightarrow v = \int \exp\left(-\int \frac{dy_1' + py_1}{y_1}dx\right) \end{aligned}$$

I shall refrain from putting this into theorem form for the sake of clarity and brevity. Instead lets do some problems,

27) Let $y = vy_1 \Rightarrow x(v''y_1 + 2v'y_1' + vy_1'') - (v'y_1 + vy_1') + 4x^3y_1 = 0$. Grouping all the v, v', and v'' terms gives,

$$xy_1v'' + 2xy_1'v' - y_1v' + \cancelto{0}{(xy_1'' - y_1' + 4x^3y_1)}v = xy_1v'' + 2xy_1'v' - y_1v' = 0.$$

Set $u = v'$, then

$$u' + \left(\frac{2y_1'}{y_1} - \frac{1}{x}\right)u = 0 \Rightarrow u' = \left(\frac{1}{x} - \frac{4x\cos x^2}{\sin x^2}\right)u = \left(\frac{1}{x} - 4x\cot x^2\right)u$$

$$\Rightarrow \ln u = \ln x - 4\int x\cot x^2 dx = \ln x - \ln\sin^2 x^2 + C \Rightarrow u = k\frac{x}{\sin^2 x^2}$$

$$\Rightarrow v = k\int \frac{xdx}{\sin^2 x^2} = k_1\cot x^2 + C \Rightarrow y = k_1\cos x^2 + C\sin x^2.$$

29) Again we let $y = vy_1 \Rightarrow x^2(v''y_1 + 2v'y_1' + vy_1'') - (x - 0.1875)vy_1 = 0$. Grouping gives,

$$x^2y_1v'' + 2x^2y_1'v' + \cancelto{0}{[x^2y_1'' - (x - 0.1875)y_1]}v = x^2y_1v'' + 2x^2y_1'v' = 0.$$

Set $u = v'$, then

$$u' = -2\frac{y_1'}{y_1}u = \left(\frac{-2}{\sqrt{x}} - \frac{1}{2x}\right)u \Rightarrow \ln u = -2\int x^{-1/2}dx + \frac{1}{2}\int\frac{dx}{x} = -4\sqrt{x} - \frac{1}{2}\ln x + C$$

$$\Rightarrow u = \frac{k}{\sqrt{x}}e^{-4\sqrt{x}} \Rightarrow v = ke^{-4\sqrt{x}} + C \Rightarrow y = kx^{1/4}e^{-2\sqrt{x}} + Cx^{1/4}e^{2\sqrt{x}}.$$

3.5 Nonhomogeneous Equations; Undetermined Coefficients

Consider the nonhomogeneous ODE,

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0 = f(x). \tag{6}$$

Notice that our usual solution wont work, but maybe it's part of the solution. Suppose y_p is the solution of (6) that is linear independent with the usual solution to the homogeneous problem. Let y be the general solution of (6). Lets plug in $y_c = y - y_p$ in (6), then we get that y_c is a solution to

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0 = 0. \tag{7}$$

So, in fact y_c is our usual homogeneous solution, so $y = y_c + y_p$, where y_c is the homogeneous part of the solution and y_p is the purely nonhomogeneous part of the solution.

Definition 1. The characteristic solution, y_c is the general solution of (7) and the particular solution, y_p is the additional solution to (6).

Case1: No term in $f(x)$ is the same as any term in y_c. Then, y_p is a linear combination of terms of $f(x)$ and their derivatives.

Ex: $f_1(x) = x^n \Rightarrow y_{1_p} = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$. If our f is a polynomial, the particular solution will be of the form of the most general polynomial of order of that of the polynomial in f.

Ex: $f_2(x) = e^{mx} \Rightarrow y_{2_p} = ke^{mx}$. This one is easy.

Ex: $f_3(x) = \cos(mx)$ or $\sin(mx) \Rightarrow y_{3_p} = A\cos(mx) + B\sin(mx)$. If we have sine or cosine our particular solution will be a linear combination of sines and cosines.

Ex: $f(x) = f_1(x) + f_2(x) + f_3(x) \Rightarrow y_p = y_{1_p} + y_{2_p} + y_{3_p}$. If we have a combination of these simple examples then we just combine all of their respective particular solutions.

Ex: $f(x) = f_1(x)f_2(x)f_3(x) \Rightarrow y_p = y_{1_p}y_{2_p}y_{3_p}$. We do the same sort of thing with products.

Case2: $f(x)$ contains terms that are x^n times terms in y_c, i.e. if $u(x)$ is a term of y_c and $f(x)$ contains $x^n u(x)$. Then y_p is as usual but multiply by "x".

Ex: Consider $y_c = g(x) + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, where we don't care about $g(x)$ and $l(x)$, we are just thinking of them as place holders. Then our particular solution is $y_p = h(x) + (A_n x^{n+1} + A_{n-1}x^n + \cdots + A_0 x)e^{mx}$.

Ex: Consider a similar case except with sine, also equivalently would be a case with cosine. $y_c = g(x) + \sin(mx)$ and $f(x) = l(x)x^n \sin(mx)$, then our particular solution is, $y_p = h(x) + (A_n x^{n+1} + A_{n-1}x^n + \cdots + A_0 x)(B\cos(mx) + C\sin(mx))$.

Case3: If y_c contains repeated roots with the highest being of order λ, i.e. x^λ, and $f(x)$ contains terms x^n times the repeated roots terms. Then multiply out by $x^{\lambda+1}$.

Ex: $y_c = g(x) + x^\lambda + \cdots + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, then our particular solution is $y_p = h(x) + x^{\lambda+1}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)e^{mx}$.

The idea for the repeated cases is to get rid of all the repeats while preserving the same amount of constants.

The cases that are delineated above are very general cases. Bellow I have set up a table of cases 2 and 3 that we will come across most often in this class. However, you may get a problem that is of a more general set up, so don't use the table as a crutch.

Case	Characteristic solution	Repeat	Particular solution form
Case2	$y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$f(x) = x^n e^{r_1 x}$	$y_p = x(A_n x^n + \cdots + A_1 x + A_0)e^{r_1 x}$
	$y_c = e^{\xi x}(A\cos(\theta x) + B\sin(\theta x))$	$f(x) = x^n e^{\xi x}\cos(\theta x)$	$y_p = x(A_n x^n + \cdots + A_0)e^{\xi x}\cos(\theta x)$
Case3	$y_c = (c_1 + c_2 x)e^{\lambda x}$	$f(x) = x^n e^{\lambda x}$	$y_p = x^2(A_n x^n + \cdots + A_1 x + A_0)e^{\lambda x}$

It can be tricky to figure out what y_p has to be at first, but hopefully some practice problems will help us,

6) We solve for the characteristic solution first, $r^2 + 2r = r(r+2) = 0 \Rightarrow r = 0, -2$, so $y_c = c_1 + c_2 e^{-2t}$, and $f(t) = 3 + 4\sin 2t$. Notice the 3 repeats with c_1. Our initial guess for the particular solution would be $y_p = A + B\cos 2t + C\sin 2t$, but this would be incorrect because we already have a lone constant in our characteristic solution, so our actual particular solution is $y_p = At + B\cos 2t + C\sin 2t$. Plugging this into the ODE gives,

$$4(C-B)\cos 2t - 4(B+C)\sin 2t + 2A = 3 + 4\sin 2t$$

Matching terms gives $2A = 3 \Rightarrow A = 3/2$ readily. From the cosine term we get $4(C-B) = 0 \Rightarrow C = B$ because there is no cosine term on the right hand side. From the sine terms we have $-4(B+C) = 8B = 4 \Rightarrow C = B = -1/2$, so our particular solution is, $y_p = \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$. Then our general solution is,

$$y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t.$$

7) As usual we find the homogeneous solution first, $r^2 + 9 = 0 \Rightarrow r = \pm 3i$, then $y_c = A\cos 3t + B\sin 3t$, and $f(t) = t^2 e^{3t} + 6$. There are no repeats, so we just proceed as usual, $y_p = (At^2 + Bt + C)e^{3t} + D$. Plugging this into the ODE gives,

$$2Ae^{3t} + 6(2At + B)e^{3t} + 18(At^2 + Bt + C)e^{3t} + 9D = t^2 e^{3t} + 6,$$
$$\Rightarrow 18At^2 e^{3t} + (12A + 18B)te^{3t} + (2A + 6B + 18C)e^{3t} + 9D = t^2 e^{3t} + 6.$$

Matching the terms readily gives $9D = 6 \Rightarrow D = 2/3$. From the $t^2 e^{3t}$ we get $18A = 1 \Rightarrow A = 1/18$. The other terms are zero so we get, $12/18 + 18B = 0 \Rightarrow B = -1/27$, and $1/9 + 2/9 + 18C = 0 \Rightarrow C = 1/162$. So, our particular solution is $y_p = (t^2/18 - t/27 + 1/162)e^{3t} + 2/3$. Then our general solution is,

$$y = A\cos 3t + B\sin 3t + \left(\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right)e^{3t} + \frac{2}{3}.$$

18) Again, $r^2 - 2r - 3 = (r-3)(r+1) = 0 \Rightarrow r = 3, -1$. This gives us a characteristic equation of $y_c = c_1 e^{3t} + c_2 e^{-t}$, and $f(t) = 3te^{2t}$. So, there are no repeats and we proceed as usual, $y_p = (At+B)e^{2t}$. Plugging this into the ODE gives,

$$4Ae^{2t} + \cancel{4(At+B)e^{2t}} - 2Ae^{2t} - \cancel{4(At+B)e^{2t}} - 3(At+B)e^{2t} = -3Ate^{2t} + (2A - 3B)e^{2t} = 3te^{2t}.$$

Matching the te^{2t} terms gives $-3A = 3 \Rightarrow A = -1$. The other term is zero so we get $-2 - 3B = 0 \Rightarrow B = -2/3$. This give us $y_p = (-t - 2/3)e^{2t}$, then our general solution is

$$y = c_1 e^{3t} + c_2 e^{-t} + \left(-t - \frac{2}{3}\right) e^{2t}.$$

The first initial condition gives, $y(0) = c_1 + c_2 - 2/3 = 1 \Rightarrow c_1 + c_2 = 5/3$, and the second gives, $y'(0) = 3c_1 - c_2 - 1 - 4/3 = 0 \Rightarrow 3c_1 - c_2 = 7/3$. Now we add the equations to get, $4c_1 = 4 \Rightarrow c_1 = 1 \Rightarrow c_2 = 2/3$. Then, our solution is,

$$y = e^{3t} + \frac{2}{3}e^{-t} + \left(-t - \frac{2}{3}\right) e^{2t}.$$

24) For this problem we only need the form of the particular solution. In order to get that we still have to compute the characteristic solution, $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$, which gives $y_c = e^{-t}(c_1 \sin t + c_2 \cos t)$. From $f(x)$ we can guess a particular solution of

$$\begin{aligned} y_p &= e^{-t}[A + B\cos t + C\sin t + (D_2 t^2 + D_1 t + D_0)\cos t + (E_2 t^2 + E_1 t + E_0)\sin t] \\ &= e^{-t}[A + (B_2 t^2 + B_1 t + B_0)\cos t + (C_2 t^2 + C_1 t + C_0)\sin t]. \end{aligned}$$

However, this would be wrong due to the repeats. So, we need to multiply out by t for the cosine and sine terms,

$$y_p = e^{-t}[A + t(B_2 t^2 + B_1 t + B_0)\cos t + t(C_2 t^2 + C_1 t + C_0)\sin t]$$