3.6 Variation of Parameters

The book does a good job at developing the theory for variation for parameters, so I will derive this in a very similar manner.

Consider the ODE,

$$y'' + p(x)y' + q(x)y = f(x), \tag{1}$$

and suppose we have the following characteristic solution,

$$y_c = c_1y_1 + c_2y_2. \tag{2}$$

What if for the full solution to (1) we can think of the "constants" c_1 and c_2 as functions, i.e. $y = u_1(x)y_1 + u_2(x)y_2$. We use this as an ansatz and plug it into the ODE. for the derivative we get,

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

We only want one derivative in our final equation so lets force

$$u_1'y_1 + u_2'y_2 = 0, \tag{3}$$

so $y' = u_1y_1' + u_2y_2'$, then

$$y'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''.$$

Plugging into (1) gives,

$$u_1'y_1' + u_2'y_2' + [u_1y_1'' + u_2y_2'' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2)] = f(x)$$

Notice the terms in brackets cancel because it is exactly the homogeneous ODE. This gives us our second equation,

$$u_1'y_1' + u_2'y_2' = f(x). \tag{4}$$

From (3) we get $u_1' = -u2'y_2/y_1$. We plug this into (4) in order to get an expression for u_2,

$$-u_2'y_1'\frac{y_2}{y_1} + u_2'y_2' = f(x) \Rightarrow -u_2'y_1'y_2 + u_2'y_2'y_1 = f(x)y_1 \Rightarrow u_2' = \frac{f(x)y_1}{y_2'y_1 - y_1'y_2} = \frac{f(x)y_1}{W(y_1, y_2)}.$$

Now we plug this into our expression for u_1 to get,

$$u_1' = \frac{f(x)y_2}{W(y_1, y_2)}.$$

Then we integrate to get,

$$u_1 = -\int \frac{f(x)y_2}{W(y_1, y_2)}dx, \tag{5}$$

$$u_2 = \int \frac{f(x)y_1}{W(y_1, y_2)}dx. \tag{6}$$

Then plugging back into our original anzats gives us,

$$y = -y_1\int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1, y_2)}dx.$$

Theorem 1. *Suppose the ODE (1) has a unique solution on I open. Assume it has the characteristic solution (2). Then,*

$$y = -y_1 \int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2 \int \frac{f(x)y_1}{W(y_1,y_2)}dx \tag{7}$$

is the general solution.

Now, we could just use this theorem for all our problems. The only downfall is that we will have to memorize this formula. So, just in case you forget the formula, do know how to work out the derivation, and try to use the derivation on specific problems.

Without further ado, lets work out some problems,

2) We go straight to the polynomial, $r^2 - r - 2 = (r-2)(r+1) = 0$, so $y_c = c_1e^{2t} + c_2e^{-t}$, so $y_1 = e^{2t}$ and $y_2 = e^{-t}$. First we calculate the Wronskian,

$$W = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t.$$

Now, lets compute our two integrals separately,

$$\int \frac{f(t)y_2}{W(y_1,y_2)}dt = \int \frac{e^{-t}\cdot 2e^{-t}}{-3e^t}dt = -\frac{2}{3}\int e^{-3t}dt = -\frac{2}{9}e^{-3t} + c_3.$$

and

$$\int \frac{f(x)y_1}{W(y_1,y_2)}dx = \int \frac{e^{2t}\cdot 2e^{-t}}{-3e^t}dt = \frac{2}{3}\int dt = -\frac{2}{3}t + c_4.$$

Then plugging this back into (7) gives,

$$y = -e^{2t}\left[-\frac{2}{9}e^{-3t} + c_3\right] + e^{-t}\left[\frac{2}{3}t + c_4\right] = \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t} + c_4e^{-t} = c_5e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t}.$$

10) Again we have, $r^2 - 2r + 1 = (r-1)^2 = 0$, so $y_c = c_1e^t + c_2te^t$, then $y_1 = e^t$ and $y_2 = te^t$, then we compute the Wronskian,

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}.$$

Now, we compute the two integrals,

$$\int \frac{f(t)y_2}{W(y_1,y_2)}dt = \int \frac{te^t \cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{tdt}{1+t^2} = \frac{1}{2}\ln(1+t^2) + c_3.$$

and

$$\int \frac{f(t)y_1}{W(y_1,y_2)}dt = \int \frac{e^t \cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{dt}{1+t^2} = \tan^{-1} t + c_4.$$

Then plugging into (7) gives,

$$y = -\frac{1}{2}e^t \ln(1+t^2) - c_3e^t + te^t \tan^{-1} t + c_4te^t.$$

11) Again, $r^2 - 5r + 6 = (r-3)(r-2) = 0$, so we get $y_c = c_1 e^{3t} + c_2 e^{2t}$, then $y_1 = e^{3t}$ and $y_2 = e^{2t}$. Taking the Wronskian gives,

$$W = \begin{vmatrix} e^3 t & e^2 t \\ 3e^3 t & 2e2^t \end{vmatrix} = -e^{5t}.$$

Then we plug it in to (7),

$$y = -e^{3t}\int \frac{e^{2t}g(t)}{-e^{5t}}dt + e^{2t}\int \frac{e^{3t}g(t)}{-e^{5t}}dt = e^{3t}\int e^{-3t}g(t)dt - e^{2t}\int e^{-2t}g(t)dt$$

Another way to write this, which will be the usual way in the book is,

$$y = c_1 e^{3t} + c_2 e^{2t} + e^{3t}\int^t e^{-3\tau}g(\tau)d\tau - e^{2t}\int^t e^{-2\tau}g(\tau)d\tau$$

14) We must first convert this into standard form,

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t.$$

First we fine the Wronskian,

$$W = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2 e^t$$

Then we plug in to (7) to get,

$$y = -t\int \frac{te^t \cdot 2t}{t^2 e^t}dt + te^t \int \frac{t \cdot 2t}{t^2 e^t}dt = -t\int 2dt + te^t \int 2e^{-t}dt = -2t^2 + c_1 t - 2t + c_2 te^t.$$

So the particular solution is,

$$y_p = -2t^2 - 2t.$$

20) We convert this to standard form,

$$y'' + \frac{1}{x}y' + \frac{x^2 - 0.25}{x^2}y = \frac{g(x)}{x^2}.$$

We find the Wronskian first,

$$W = \begin{vmatrix} x^{-1/2}\sin x & x^{-1/2}\cos x \\ -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x & -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x \end{vmatrix} = -\frac{1}{x}$$

Then plugging into (7) gives,

$$y = -x^{-1/2}\sin x \int \frac{x^{-1/2}\cos x g(x)/x^2}{-1/x}dx + x^{-1/2}\cos x \int \frac{x^{-1/2}\sin x g(x)/x^2}{-1/x}dx$$
$$= x^{-1/2}\sin x \int \frac{\cos x g(x)}{x\sqrt{x}}dx - x^{-1/2}\cos x \int \frac{\sin x g(x)}{x\sqrt{x}}dx$$

Then the particular solution is,

$$y_p = x^{-1/2}\sin x \int^x \frac{\cos \xi g(\xi)}{\xi\sqrt{\xi}}d\xi - x^{-1/2}\cos x \int^x \frac{\sin \xi g(\xi)}{\xi\sqrt{\xi}}d\xi$$

3.7 Applications: Mechanical and Electrical Oscillators

These are the sort of examples we will deal with in our Chaos course, except of course the Chaos example are going to be much more difficult, and with forcing involved.

Consider a mass on a weightless-hanging spring. Gravity balances with the spring force for that particular distance, so we can neglect it. All we need are the additional forces on the system.

The total force on the entire system is mx''. The spring force is kx, and the retarding force is $\gamma x'$. Any external force is neglected in this section, but if it were not it would just be $F(t)$. Applying Newton's laws gives,

$$mx'' = -kx - \gamma x' \Rightarrow mx'' + \gamma x' + kx = 0; x(0) = x_0, x'(0) = v_0. \tag{8}$$

We also have to be aware of the units, m = [mass], γ = [mass/time], and k = [mass/time^2].

We look at a few cases.

Undamped: Here $\gamma = 0$, so our equation becomes,

$$mx'' + kx = 0. \tag{9}$$

This has an easy solution,

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t. \tag{10}$$

Here $\omega = \sqrt{k/m}$ is called the natural frequency. Now, lets think of this in the complex plane and try to determine some important quantities. Think of ξ being the x-axis and η being the y-axis in the complex plane. And define the unit vectors to be $\hat{\xi} = \cos\omega t$ and $\hat{\eta} = \sin\omega t$. Then we can draw a triangle where the side on the x-axis is of length A and the side on the y-axis is of length B. Also let the angle adjacent to the x-axis be called ϕ. Then we have that the hypotenuse, $R = \sqrt{A^2 + B^2}$, which is the amplitude of oscillation, and the angle ϕ called the phase, is given by $\tan\phi = B/A$. We can also use this triangle to simplify our equation. Notice that $A = R\cos\phi$ and $B = R\sin\phi$. Then, by using trig identities,

$$x = R\cos\phi\cos\omega t + B\sin\phi\sin\omega t = R\cos(\omega t - \phi).$$

Also notice that $x(0) = R\cos\phi$. When does $x = R\cos\phi$ again? We can show that this happens at every addition of $2\pi/\omega$, so our period is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.

Damped: Now we explore what happens when we have damping. This gives rise to three cases. Here we will have the full ODE, so our roots of the characteristic polynomial is,

$$r = \frac{1}{2m}\left(-\gamma \pm \sqrt{\gamma^2 - 4mk}\right).$$

If $\gamma^2 - 4mk > 0$, our solution becomes $x = c_1e^{r_1x} + c_2e^{r_2x}$, and this is called overdamped, because it goes to zero very fast.

If $\gamma^2 - 4mk = 0$, our solution becomes $x = (c_1 + c_2t)e^{rt}$, where r is a repeated root, and this is called critically damped because after some critical point it damps to zero very fast.

If $\gamma^2 - 4mk < 0$, our solution becomes $x = e^{\xi t}(A\cos\theta t + B\sin\theta t)$. This is a bit of a special case. If $\xi = -\gamma/2m$ is large it acts like the preceding case, if it's small then we get a special type of behavior called underdamped motion. This is because the system will oscillate while damping out. Here θ is called the quasi frequency and $\theta/\omega < 1$. Similarly, $T_d = 2\pi/\theta$ is called the quasi period and $T_d/T > 1$.

These cases are outlined in the following handy-dandy table,

Type	Criterion	Solution
Undamped	$\gamma = 0$	$x = A\cos\omega t + B\sin\omega t$
Overdamped	$\gamma^2 - 4mk > 0$	$x = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
Critically Damped	$\gamma^2 - 4mk = 0$	$x = (c_1 + c_2 t)e^{rt}$
Underdamped	$\gamma^2 - 4mk < 0$	$x = e^{\xi t}(A\cos\theta t + B\sin\theta t)$

The next table outlines oscillatory behavior,

Type	Criterion	Solution	Frequency	Period
Undamped	$\gamma = 0$	$x = A\cos\omega t + B\sin\omega t$	$\omega = \sqrt{k/m}$	$T = 2\pi/\omega$
Underdamped	$\gamma^2 - 4mk < 0$	$x = e^{\xi t}(A\cos\theta t + B\sin\theta t)$	$\theta = (\sqrt{4mk - \gamma^2})/2\gamma$	$T_d = 2\pi/\theta$

Now lets do a couple of spring problems,

4) We calculate the amplitude in the usual manner, $R = \sqrt{4+9} = \sqrt{13}$. And the phase, which they call δ and we call ϕ, $\tan\delta = -2/-3$, which means we are in the third quadrant, so $\delta = \tan^{-1}\left(\frac{2}{3}\right) + \pi$. Finally, the frequency is $\omega = \pi$.

6) Here we must first calculate the spring constant. Recall Hooke's law, $F = kx \Rightarrow k = F/x = (.1)(9.8)/.05 = 19.6\text{N/m}$. Since there is no retarding force our ODE is,

$$mx'' + kx = 0;\ x(0) = 0,\ x'(0) = .1,$$

which has a general solution of,

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}} = A\cos 14t + B\sin 14t.$$

From the initial conditions we get, $A = 0$, $14B = .1 \Rightarrow B = 1/140$. This part is done incorrectly in the book because they forgot to be consistent with the units. So, our solution is,

$$x = \frac{1}{140}\sin 14t.$$

This means that the time of return is $t_1 = \pi/14$, and the period is $T = \pi/7$.

We can look at another application: The RLC circuit. Earlier in the semester we studied the RL circuit, which had an ODE of $LdI/dt + RI = V$. Now, for the RLC circuit, the voltage across the conductor is, $Q(t)/C = (1/C)\int_{t_0}^{t} I(\tau)d\tau + V_C(t_0)$, then our ODE becomes,

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_{t_0}^{t} I(\tau)d\tau + V_C(t_0) = V. \tag{11}$$

Now, we can differentiate through to get,

$$LI'' + RI' + \frac{1}{C}I = V';\ I(t_0) = I_0,\ I_0'(t_0) = \frac{1}{L}(V(t_0) - RI_0 - Q_0/C). \tag{12}$$

However recall, $I = dQ/dt$, so we can plug this into (11) to get,

$$LQ'' + RQ' + \frac{1}{C}Q = V;\ Q(t_0) = Q_0,\ Q'(t_0) = I(t_0) = I_0. \tag{13}$$

Notice that the equations are just like the spring equations.

Here is an example of an electrical problem,

8) For this problem we use (13). We discharge the capacitor without incoming voltage and there is no resistor, so our equation is, $LQ'' + Q/C = 0$, but since $L = 1$, it's easier just to plug this in straight away, so $Q'' + Q/C = 0$. Solving the roots for this gives, $r^2 + 1/C = 0$, then $r = \pm i\sqrt{1/C}$, then our general solution is,

$$Q = A\cos\sqrt{1/C}t + B\sin\sqrt{1/C}t.$$

Plugging in the initial conditions gives, $Q(0) = A = 10^{-6}$, and $Q'(0) = B\sqrt{1/C} = 0 \Rightarrow B = 0$. So, our solution is,

$$Q(t) = 10^{-6}\cos(2\times 10^3)t.$$

Now, lets do a few more theoretical problems for oscillators in general,

13) Notice, for this problem $\omega = \sqrt{\frac{k}{m}} = 1$. Now, we solve the full ODE, which gives $r^2 + \gamma r + 1 = 0$, which gives us roots of, $r = \frac{1}{2}(-\gamma) \pm \sqrt{\gamma^2 - 4} = \frac{1}{2}(-\gamma) \pm i\sqrt{4-\gamma^2}$. This means that $\theta = \sqrt{4-\gamma^2}/2$.

19) Lets first look at the more difficult case: overdamped. Then our solution is $x = c_1e^{r_1t} + c_2e^{r_2t};\ r_{1,2} \le 0$. Now, if we want to see when the mass crosses the origin we set $x = c_1e^{r_1t} + c_2e^{r_2t} = 0$, which means

$$c_1e^{r_1t} = -c_2e^{r_2t} \Rightarrow -\frac{c_1}{c_2} = e^{(r_2-r_1)t} \Rightarrow \ln\left(-\frac{c_1}{c_2}\right) = (r_2-r_1)t \Rightarrow t = \frac{1}{r_2-r_1}\ln\left(-\frac{c_1}{c_2}\right).$$

This shows that the mass crosses the origin at most once.

Now, lets do the easier case: critically damped. Then our solution is $x = (c_1 + c_2t)e^{rt}$, so if $x = 0$, $c_1 + c_2t = 0 \Rightarrow t = -c_1/c_2$, which leads us to our conclusion.