## 6.2 IVPs WITH LAPLACE TRANSFORMS

I'll "handwave" because deeper knowledge is required to properly understand the theory, which exceeds the scope of this course.

In order to apply Laplace Transforms to ODEs we have to take the Laplace Transform of derivatives. Let $Y = \mathcal{L}\{y\}$ and $y' = dy/dt$, then

$$\mathcal{L}\{y'\} = \int_0^\infty e^{-st}y'dt = e^{-st}y|_0^\infty + s\int_0^\infty e^{-st}ydt = -y(0) + sY. \qquad (1)$$

We can get higher derivatives by induction,

$$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\} = -y'(0) + s\mathcal{L}\{y'\} = -y'(0) - sy(0) + s^2Y. \qquad (2)$$

Similar calculations can be found for higher derivatives.

Lets look at some examples,

5) We first recognize what it resembles and try to convert it into that form,

$$\frac{2s+2}{s^2+2s+5} = 2\frac{s+1}{(s+1)^2+4} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos 2t.$$

8) First we do the partial fractions,

$$\frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow As^2 + 4A + Bs^2 + Cs = (A+B)s^2 + Cs + 4A = 8s^2 - 4s + 12.$$

Then we get $A = 3$, $C = -4$, and $B = 5$, then

$$F(s) = \frac{3}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4} \Rightarrow \mathcal{L}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t.$$

14) We take the Laplace Transform of the entire ODE,

$$-y'(0)^{\nearrow 1} - sy(0)^{\nearrow 1} + s^2Y + 4y(0)^{\nearrow 1} - 4sY + 4Y = 0 \Rightarrow (s^2 - 4s + 4)Y = s - 3 \Rightarrow Y = \frac{s-3}{s^2-4s+4}$$

$$\Rightarrow Y = \frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2} \Rightarrow y(t) = e^{2t} - te^{2t}.$$

22) Again we take the Laplace Transform of the entire ODE,

$$-y'(0)^{\nearrow 1} - sy(0)^{\nearrow 0} + s^2Y + 2y(0)^{\nearrow 0} - 2sY + 2Y = \frac{1}{s+1} \Rightarrow (s^2 - 2s + 2)Y = \frac{1}{s+1} + 1$$

$$\Rightarrow Y = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}.$$

The second term is fine, but for the first time we must do partial fractions,

$$\frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} \Rightarrow As^2 - 2As + 2A + Bs^2 + Bs + Cs + C = (A+B)s^2 + (B+C-2A)s + 2A + C = 1.$$

Then we get $A = -B \Rightarrow C = 3A$, then $A = 1/5 = -B$, and $C = 3/5$.

Hence,

$$Y = \frac{1}{5} \cdot \frac{1}{s+1} + \frac{1}{5} \cdot \frac{-s+8}{s^2-2s+2} = \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + 7\frac{1}{(s-1)^2+1}$$

$$\Rightarrow y = \frac{1}{5}\left[e^{-t} - e^t\cos t + 7e^t\sin t\right].$$

Lets start with an example from the previous section:

Sec 6.2 #24: Lets take the Laplace Transform for the discontinuous function using the definition,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^\pi e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\pi = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Now we put this back into the Laplace Transform for the entire ODE,

$$-y'(0) - sy(0) + s^2 Y + 4Y = \frac{1}{s} - \frac{e^{-\pi s}}{s} \Rightarrow (s^2+4)Y = \frac{1}{s} - \frac{e^{-\pi s}}{s} + s \Rightarrow Y = \frac{1}{s(s^2+4)} - \frac{e^{-\pi s}}{s(s^2+4)} + \frac{s}{s^2+4}.$$

We actually solved this in class, but lets skip that here.

This leads to the very natural question: What happens when our forcing function (i.e. Right Hand Side (RHS)) is discontinuous? This is when step functions come in. Lets first recall some definitions from Calc I and also develop some theory about the Laplace Transform of step functions.

Recall the definition for a step function,

$$u_c(t) = \begin{cases} 0, & \text{for } t < c, \\ 1, & \text{for } t \geq c; \end{cases} \tag{3}$$

Lets find the Laplace Transform of this,

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{1}{s} e^{-cs}. \tag{4}$$

Now lets consider a forcing function where instead of one there is some variable forcing $f(t)$ after time $t = c$, then we have that the forcing is $f(t-c)u_c(t)$. Lets find the Laplace Transform of this,

$$\mathcal{L}\{f(t-c)u_c(t)\} = \int_0^\infty e^{-st} f(t-c) u_c(t) dt = \int_c^\infty e^{-st} f(t-c) dt = \int_0^\infty e^{-s(\tau+c)} f(\tau) d\tau$$

$$= e^{-cs} \int_0^\infty e^{-s\tau} f(\tau) d\tau = e^{-sc} \mathcal{L}\{f(\tau)\} = e^{-cs} F(s). \tag{5}$$

Now lets do some problems,

3) Here $g(t) = (t - \pi)^2 u_\pi(t)$, and the plot is the one on the left,
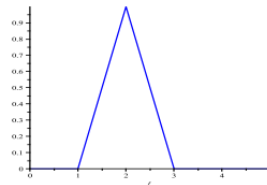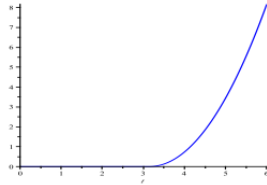


6) Here the equation is and the plot is the one on the right. $g(t) = \begin{cases} 0 & t < 1, \\ t - 1 & 1 \leq t < 2, \\ 3 - t & 2 \leq t < 3, \\ 0 & 3 \leq t; \end{cases}$

13) Here $f(t) = (t-2)^2 u_2(t)$ and $\mathcal{L}\{t^2\} = 2/s^3$, then $\mathcal{L}\{f(t)\} = 2e^{-2s}/s^3$.

18) Here $\mathcal{L}\{t\} = 1/s^2$, so $\mathcal{L}\{f(t)\} = 1/s^2 - e^{-s}/s^2$.

22) Here $c = 2$, so $G(s) = 2/(s^2 - 2^2)$, then $g(t) = \sinh 2t$, hence $f(t) = u_2(t) \sinh(2(t-2))$.

We discussed discontinuous forcing last time. Lets now do a bunch of problems,

1) Here $f(t) = 1 - u_{3\pi}(t)$, then the Laplace Transform of the full ODE is,

$$-y'(0)^{\nearrow 1} - sy(0)^{\nearrow 0} + s^2 Y + Y = \frac{1}{s} - \frac{1}{s}e^{-3\pi s} \Rightarrow (s^2 + 1)Y = 1 + \frac{1}{s} - \frac{1}{s}e^{-3\pi s}$$

$$\Rightarrow Y = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)}e^{-3\pi s} \Rightarrow Y = \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} - \left(\frac{1}{s^2+1} + \frac{1}{s}\right)e^{-3\pi s}$$

$$\Rightarrow y = \sin t + 1 - \cos t - [1 - \cos(t-3\pi)]u_{3\pi}(t) = \sin t + 1 - \cos t - [1 + \cos t]u_{3\pi}(t)$$

4) We take the Laplace Transform of the full ODE,

$$-y'(0)^{\nearrow 0} - sy(0)^{\nearrow 0} + s^2 Y + 4Y = \frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-\pi s} \Rightarrow Y = \frac{1}{(s^2+4)(s^2+1)} + \frac{1}{(s^2+4)(s^2+1)}e^{-\pi s}$$

$$\Rightarrow Y = \frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+2^2} + \left(\frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+2^2}\right)e^{-\pi s}$$

$$\Rightarrow y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \left(\frac{1}{3}\sin(t-\pi) - \frac{1}{2}\sin(2t-2\pi)\right)u_\pi(t).$$

6) We take the Laplace Transform,

$$-y'(0)^{\nearrow 1} - sy(0)^{\nearrow 0} + s^2 Y - 3y(0)^{\nearrow 0} + 3sY + 2Y = \frac{1}{s}e^{-2s} \Rightarrow (s^2+3s+2)Y = 1 + \frac{1}{s}e^{-2s}$$

$$\Rightarrow Y = \frac{1}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}e^{-2s} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right)e^{-2s}$$

$$\Rightarrow y = e^{-t} - e^{-2t} + \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)u_2(t)$$

8) Taking the Laplace Transform gives,

$$-y'(0)^{\nearrow 0} - sy(0)^{\nearrow 0} + s^2 Y - y(0)^{\nearrow 0} + sY + \frac{5}{4}Y = \frac{1}{s^2} - \frac{1}{s^2}e^{-\pi s/2} \Rightarrow (s^2+s+5/4)Y = \frac{1}{s^2} - \frac{1}{s^2}e^{-\pi s/2} \Rightarrow Y = \frac{1}{s^2(s^2+s+5/4)} - \frac{1}{s^2(s^2+s+5/4)}e^{-\pi s/2}.$$

Now we do the partial fractions,

$$\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+s+5/4} \Rightarrow As^3 + As^2 + \frac{5}{4}As + Bs^2 + Bs + \frac{5}{4}B + Cs^3 + Ds^2 = (A+C)s^3 + (A+B+D)s^2 + (5A/4+B)s + \frac{5}{4}B = 1.$$

Then we get, $B = 4/5$, $A = -16/25$, $C = 16/25$, and $D = -4/25$, then

$$Y = \left[-\frac{16}{25}\cdot\frac{1}{s} + \frac{4}{5}\cdot\frac{1}{s^2} + \frac{16}{25}\cdot\frac{s-1/4}{s^2+s+5/4}\right]\left(1 - e^{-\pi s/2}\right).$$

That final term in the brackets is going to be the tough,

$$\frac{s-1/4}{s^2+s+5/4} = \frac{s-1/4}{s^2+s+1/4+1} = \frac{s+1/2}{(s+1/2)^2+1} - \frac{3/4}{(s+1/2)^2+1}.$$

Then the solution is,

$$y = \frac{16}{25}\left(e^{-t/2}\cos t - \frac{3}{4}e^{-t/2}\sin t + \frac{5}{4}t - 1\right) - \frac{16}{25}u_{\pi/2}(t)\left(e^{-(t-\pi/2)/2}\cos(t-\pi/2) - \frac{3}{4}e^{-(t-\pi/2)/2}\sin(t-\pi/2) + \frac{5}{4}(t-\pi/2) - 1\right).$$