7.2 Matrices and 7.3 Eigenvalues/Eigenvectors

We went through these kind of quickly, but know how to multiply vectors by matrices, take the determinant, find whether or not vectors are linearly independent (similar to the Wronskian), find eigenvalues and eigenvectors. Also you only have to worry about 2x2 matrices.

7.5 Homogeneous linear systems with constant coefficients

These are basically like our second order problems, we just have to take the eigenvalue of the matrix and treat them as our roots. Then we compute the eigenvectors. Recall we find the eigenvalues by computing what values of λ satisfy $\det(A-\lambda I)=0$. And the eigenvectors are found by computing the values of x that satisfy $(A-\lambda I)x=0$.

1) The eigenvalues are,

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+4 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = -1,\, 2$$

Now we compute the eigenvectors,

$$\begin{pmatrix} 3-\lambda_1 & -2 \\ 2 & -2-\lambda_1 \end{pmatrix} x^{(1)} = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} x^{(1)} = 0 \Rightarrow x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3-\lambda_2 & -2 \\ 2 & -2-\lambda_2 \end{pmatrix} x^{(2)} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} x^{(2)} = 0 \Rightarrow x^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then our solution becomes,

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

Now, if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

3) Again we find the eigenvalues,

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+3 = \lambda^2-1 = 0 \Rightarrow \lambda = \pm 1.$$

Then the eigenvectors,

$$x^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\; x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Here if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

5) Again,

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = (\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = -1,\, -3.$$

And the eigenvectors,

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Our solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

Here $x \to 0$.

8) Again the eigenvalues are,

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+6 = -6-\lambda+\lambda^2+6 = \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0,\, 1.$$

with the eigenvectors,

$$x^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Then our solution is,

$$x = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

So our solution behaves as follows: if $c_2 = 0$, $x = c_1(-2, 1)$ i.e. the first eigenvector. If $c_2 \neq 0$, $x \to \infty$.

7.6 Complex eigenvalues

This works in a similar manner to the second order equations, except we can't get a nice closed form formula. Therefore, we have to do some intermediate steps before we get our solution. Lets look at how we derive this method with the first problem, then after that we will skip some steps.

3) We take the eigenvalues as usual,

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (4-\lambda^2) + 5 = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i.$$

Then the eigenvectors are,

$$x^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

Then our solution is

$$\hat{x} = c_1 \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-it} = c_1 \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) + c_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} (\cos t - i \sin t)$$

$$= c_1 \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \right] + c_2 \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t - 2\sin t \\ -\sin t \end{pmatrix} \right]$$

So, our real solution would be,

$$x = (c_1 + c_2) \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + (c_1 - c_2) \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Notice, since the eigenvectors are complex conjugates, we only need one eigenvector to find our solution. This is what we will do from now on.

9) Here we'll do an IVP. We take the eigenvalues,

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = -3+2\lambda+\lambda^2+5 = \lambda^2+2\lambda+2 = 0 \Rightarrow \lambda = \frac{1}{2}(-2\pm\sqrt{4-8}) = -1\pm i.$$

The eigenvector for $\lambda = -1+i$ is,

$$x^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \Rightarrow \hat{x} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i\sin t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Then our general solution is,

$$x = Ae^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + Be^{-t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Now we plug in our initial conditions,

$$x(0) = A\begin{pmatrix} 2 \\ 1 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow A = 1 \Rightarrow B = -1.$$

Then our solution is,

$$x = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

15) Here we consider what happens as a parameter α changes. We can do this purely through the eigenvalues,

$$\begin{vmatrix} 2-\lambda & -5 \\ \alpha & -2-\lambda \end{vmatrix} = -(4-\lambda^2) + 5\alpha = \lambda^2 + (5\alpha - 4) = 0 \Rightarrow \lambda = \pm\sqrt{4-5\alpha}.$$

Notice that $\lambda = 0$ when $\alpha_* = 4/5$, so this is the critical value. Now, when $\alpha < \alpha_*$ we get a saddle and when $\alpha > \alpha_*$ we get centers. You wont have to know how to do phase planes on the exam.