10.1 Boundary Value Problems

We are used to initial value problems where we are given initial data. What if we are given boundary data instead? There are many applications where things are happening for a long period of time and we don't know what happened in the beginning, but we do know something about the boundary. The usual problems are solved in a similar fashion to Initial Value Problems. We do however have a bit more theory.

Definition 1. The boundary values (for a second order ODE) $y(a)$, $y(b)$, $y'(a)$, and/or $y'(b)$ are said to be homogeneous if any two of the above boundary data are zero.

We also have eigenvalue problems for BVPs. Recall that for matrices the eigenvalue problems were of the form $Ax = \lambda x$, where we solve for the "eigenvalue", λ. For BVPs of a second order ODE, we consider our linear operator to be $L = d^2/dx^2$ (for matrices the linear operator is the matrix A). So we wish to solve the problem $Ly = \lambda y$, i.e. $y'' + \lambda y = 0$. Here the $y_n's$ corresponding to $\lambda_n's$ are called eigenfunctions (similar to eigenvectors in the matrix case). We notice that eigenvalue problems are only for homogeneous boundary values.

Definition 2. The boundary value problem,

$$y'' + \lambda y = 0; \text{ (with homogeneous boundary values)}, \tag{1}$$

is called an eigenvalue problem. And the nontrivial (i.e. $y_n \neq 0$) solutions y_n corresponding to λ_n are the eigenfunctions of the corresponding eigenvalues.

Now, lets do some regular problems,

4) The characteristic polynomial gives us,

$$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y = A\cos t + B\sin t \Rightarrow y' = -A\sin t + B\cos t.$$

Then our first boundary value gives, $y'(0) = B = 1$, and

$$y(L) = A\cos L + \sin L = 0 \Rightarrow A = -\tan L \quad \text{if } L \neq (2k+1) * \pi/2 \text{ for } k = 0, \pm 1, \pm 2, \ldots$$

However, if $\cos L = 0$, $\sin L = 0$, but this is clearly false because $\sin x \neq 0$ when $\cos x = 0$ and vice-versa, so the BVP has no solution.

6) Again we use the characteristic polynomial,

$$r^2 + 2 = 0 \Rightarrow r = \pm i\sqrt{2} \Rightarrow y_c = A\cos\sqrt{2}x + B\sin\sqrt{2}x.$$

Recall we need to find the particular solution. Guess the form, $y_p = c_1x + c_2$ and plug this into the ODE to get $2(c_1x + c_2) = x \Rightarrow c_2 = 0$, $c_1 = 1/2$, then our general solution is $y = A\cos\sqrt{2}x + B\sin\sqrt{2}x + x/2$. The first initial condition gives, $y(0) = A = 0$ and the second gives,

$$y(\pi) = B\sin\sqrt{2}\pi + \frac{\pi}{2} = 0 \Rightarrow B = \pi/(2\sin\sqrt{2}\pi) \Rightarrow y = -\frac{\pi}{2}\left(\frac{\sin\sqrt{2}x}{\sin\sqrt{2}\pi}\right) + \frac{x}{2}.$$

9) Again,

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y_c = A\cos 2x + B\sin 2x.$$

Again we look for a particular solution of the form, $y_p = C\cos x + D\sin x$,

$$-C\cos x - D\sin x + 4C\cos x + 4D\sin x = 3C\cos x + 3D\sin x = \cos x \Rightarrow D = 0,\ C = \frac{1}{3}.$$

Then our general solution is

$$y = A\cos 2x + B\sin 2x + (1/3)\cos x \Rightarrow y' = -2A\sin 2x + 2B\cos 2x - \frac{1}{3}\sin x.$$

The first boundary condition gives, $y'(0) = 2B = 0 \Rightarrow B = 0$. For the second one we get, $y'(\pi) = 0$ trivially, so we will get infinitely many solutions,

$$y = A\cos 2x + \frac{1}{3}\cos x.$$

Now we'll do a couple of eigenvalue problems. Recall for these types of problems we have to actually do three separate problems for $\lambda > 0$, $\lambda < 0$, and $\lambda = 0$. Also remember that we don't want trivial solutions.

16) **(i)** If $\lambda > 0$, let $\lambda = \mu^2$. Then,

$$r = \pm i\mu \Rightarrow y = A\cos\mu t + B\sin\mu t \Rightarrow y' = -A\mu\sin\mu t + B\mu\cos\mu t.$$

From the first boundary condition we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ because $\lambda > 0$. From the second B.C. we get $y'(\pi) = -A\mu\sin\mu\pi = 0$. Since we don't want trivial solutions we can't have $A = 0$, so we require $\sin\mu\pi = 0$ then $\mu = n\pi$ where $n = 1, 2, \ldots$, so our eigenfunctions for corresponding eigenvalues are,

$$y_n = \cos n\pi t;\ \lambda_n = n^2,\ n \in \mathbb{N} \Leftrightarrow n = 0, 1, 2, \ldots$$

(ii) If $\lambda < 0$, let $\lambda = -\mu^2$. Then,

$$r = \pm\mu \Rightarrow y = c_1 e^{\mu t} + c_2 e^{-\mu t} = A\cosh\mu t + B\sinh\mu t \Rightarrow y' = A\sinh\mu t + B\cosh\mu t.$$

The B.C's give, $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y'(\pi) = A\sinh\mu\pi = 0$, so either $\mu = in\pi$ or $A = 0$. Since we don't want complex solutions, we have $A = 0$, then $y = 0$, so unfortunately we get a trivial solution.

(iii) If $\lambda = 0$, $y = c_1 + c_2 x \Rightarrow y' = c_2$, then applying the B.C's give $y'(0) = c_2 = 0$ and $y'(\pi) = 0$ automatically. So our eigenvalue and eigenfunction are,

$$y_0 = 1, \lambda_0 = 0.$$

Notice I left out the constants. It is up to you whether or not you would like to keep the constants there or leave them out.

17) **(i)** If $\lambda > 0$, let $\lambda = \mu^2$, then

$$y = A\cos\mu t + B\sin\mu t \Rightarrow y' = -A\mu\sin\mu t + B\mu\cos\mu t.$$

Notice, how we have the same exact general solution! You do enough of these problems and you can go straight to the solution and it's derivative without having to do the characteristic polynomial. Now, from the B.C's we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y(L) = A\cos\mu L = 0$. So, we require $\mu = (2n-1)\pi/2L$ where $n \in \mathbb{N}$, then our eigenvalues and eigenfunctions are,

$$y_n = A\cos\left((2n-1)\frac{\pi}{2}t\right)\text{ ; } \lambda_n = (2n-1)^2\frac{\pi^2}{4},\ n \in \mathbb{N}$$

(ii) If $\lambda < 0$, let $\lambda = -\mu^2$, then

$$y = A\cosh\mu t + B\sinh\mu t \Rightarrow y' = A\mu\sinh\mu t + B\mu\cosh\mu t.$$

From the B.C's we get $y'(0) = B\mu = 0 \Rightarrow B = 0$ and $y(L) = A\cosh\mu L = 0 \Rightarrow A = 0$, again it's a trivial solution $y = 0$.

(iii) If $\lambda = 0$, $y = c_1 + c_2x \Rightarrow y' = c_2$, from the B.C's we get $y'(0) = c_2 = 0$ and $y(L) = c_1 = 0$, so again we have a trivial solution, $y = 0$.

10.2 Fourier Series

For these spaces we can find many analogues to vector spaces. The difference is that now we are thinking in terms of infinite dimensional spaces. In order to ease the transition I will provide analogues for most definitions.

Definition 3. We call the functional,

$$< u, v >= \int_\alpha^\beta u(x)v(x)dx, \tag{2}$$

the L^2 inner product on the interval $[\alpha, \beta]$.

This is similar to the dot product.

Definition 4. If $< u, v >= 0$ we say u and v are orthogonal.

This is like perpendicular vectors. Recall two vectors a and b are perpendicular if $a \cdot b = 0$.

Definition 5. We call $||u|| =< u, u >$ the L^2 norm of u.

Definition 6. If $||u|| = 1$ we say u is normal.

Definition 7. If $||u|| = 1$ and $||v|| = 1$ and $< u, v >= 0$ we say that u and v are orthonormal.

Now equipped with our new machinery we can derive a series approximation that is ideal for periodic functions. We did this in class, but here I shall just remind you of the formulas:

Fourier Series

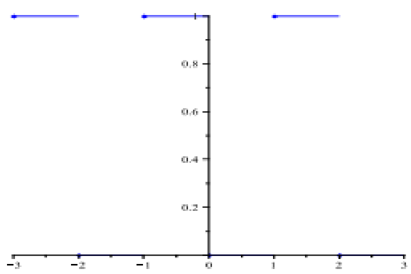
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{3}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \; a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Now lets do some problems,

2) $T = 1$ 3) Not periodic 4) $T = 2L$ 8) $T = 4$.



14) (a) This is what the plot looks like if $L = 1$, but we should keep things general on the exam.

(b) We first do a_0,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{0} dx = 1.$$

Then for a_n,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} \rightarrow 0$$

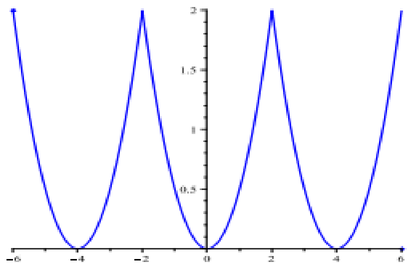
Finally, for b_n,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0}$$

$$= -\frac{1}{n\pi} + \frac{1}{n\pi}\cos(n\pi) = \frac{-1+(-1)^n}{n\pi} = -\frac{2}{n\pi}\begin{cases} 1 & n \text{ odd, i.e. } n = 2k+1;\; k = 0, \pm 1, \pm 2, \ldots, \\ 0 & n \text{ even, i.e. } n = 2k;\; k = 0, \pm 1, \pm 2, \ldots; \end{cases}$$

Then our Fourier series becomes,

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin\left(\frac{1}{L}(2k+1)\pi x\right).$$

21) (a)  This is what the plot looks like.

(b) We do a_0 first,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}dx = \frac{x^3}{12}\bigg|_{-2}^{2} = \frac{4}{3}.$$

Now to do a_n we need to do by parts twice, the details of which were shown in class, but here I'll just give you the final form,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx = \int_{0}^{2}\frac{x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{(n\pi)^3}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 = \frac{8}{(n\pi)^2}\cos(n\pi) = (-1)^n\frac{8}{(n\pi)^2}.$$

For b_n we get,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

because we are integrating an odd function on a symmetric interval. Then our Fourier series is,

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

10.4 Even and Odd Functions

As we saw for the last problem in the preceding section, it can be useful to know whether or not a function is odd or even. Also, many times we will want the Fourier series of a non-periodic function. In order to do this we need to create a periodic function that includes our non-periodic function. Instead of creating something that is neither odd nor even if we create an even or odd function we can save a lot of time.

Definition 8. Consider a function $f(x)$ such that $f(-x) = f(x)$, then f is said to be even.

Definition 9. Consider a function $f(x)$ such that $f(-x) = -f(x)$, then f is said to be odd.

There are some important properties that we should keep in mind,

Properties

- Sum/difference of two even functions is even.
- Sum/difference of two odd functions is odd.
- sum/difference of and even and an odd function is neither even nor odd.
- Product/quotient of two even functions is even.
- Product/quotient of two odd functions is even.
- Product/quotient of an even and an odd function is odd.
- If f is even, $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.
- If f is odd, $\int_{-L}^{L} f(x)dx = 0$.

Now we can think of a Fourier Cosine series and Fourier Sine series. These can be derived straight from the Fourier series equations so it's best not to memorize these formulas.

Fourier Cosine Series: If f is an even periodic function generated on $-L \leq x \leq L$, then $b_n = 0$, so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right)\right] \tag{4}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx,\ a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx;$$

Fourier Sine Series: If f is an odd periodic function generated on $-L \leq x \leq L$, then $a_n = 0$, so

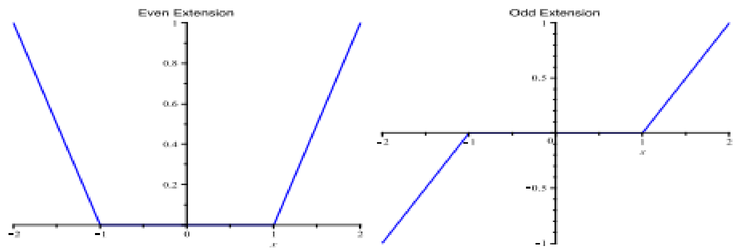
$$f(x) = \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{5}$$

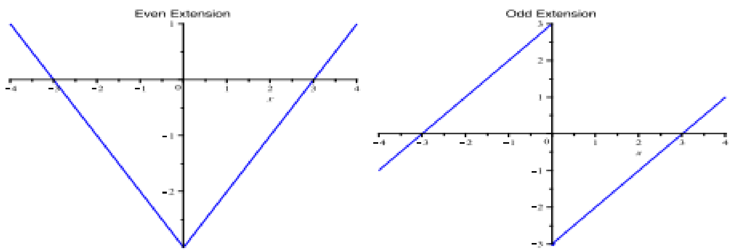
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx;$$

Now lets do some simple problems,

1) Odd 5) Even 6) Neither
For each of these we just apply the definitions.

Periodic Extensions. Suppose a function f is defined only on $[0, L]$. If we want to find the Fourier Series of this we need to make a periodic function that "includes" f. These are called periodic extensions and can either be odd or even.

8) 

10) 

22) (a) Notice that for odd extensions our periodic function of period $2L$ becomes,

$$g(x) = \begin{cases} -f(-x) & -L < x < 0, \\ f(x) & 0 < x < L; \end{cases} = \begin{cases} -L - x & -L < x < 0, \\ L - x & 0 < x < L; \end{cases}$$

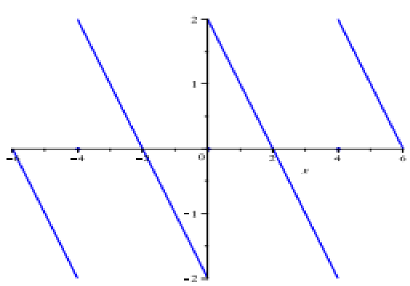
We know that for odd extensions we'll get a sine series so we only do the sine calculations,

$$b_n = \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)dx = -(L-x)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

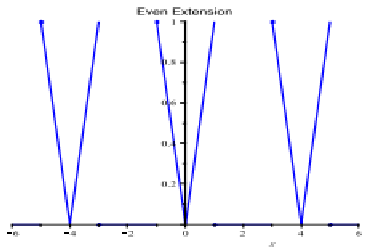
$$= \frac{2L}{n\pi} + \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L \rightarrow 0$$

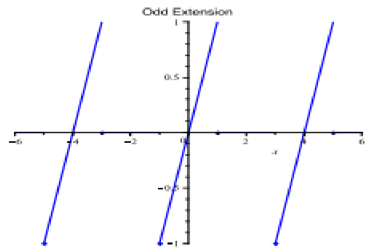
Then our Fourier series is,

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$$

(b) 

28) (a)





(b) For the Cosine series we have,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \int_0^1 x dx = \frac{1}{2}.$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}.$$

Then our Fourier series is,

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\right]\cos\left(\frac{n\pi x}{2}\right).$$

For the Sine series we have,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

Then our Fourier series is,

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right).$$