2.7 Euler's Method

Numerical solutions to ODEs are all about approximating a derivative and using that to approximate the solution. What is the definition of the derivative and how do we approximate it? Think back to Calc I, we derived the definition of the derivative by using a slope and watching what happens when $\Delta t \to 0$. Lets use the formula for slope again for *first order* ordinary differential equations,

$$y'(t) = f(t,y) \Rightarrow f(t,y) \approx \frac{\Delta y}{\Delta t} = \frac{y - y_0}{t - t_0}.$$

Now lets evaluate f at t_1, y_1, then we get,

$$f(t_1, y_1) \approx \frac{y_1 - y_0}{t_1 - t_0} \Rightarrow y_1 - y_0 \approx (t_1 - t_0) f(t_0, y_0) \Rightarrow y_1 \approx y_0 + (t_1 - t_0) f(t_0, y_0).$$

Look at that! We just developed a formula to approximate y at t_1 by using the information we had for the system at t_0. If we can approximate the data at t_1 by using the previous time (i.e. t_0), why can't we do this for any time? That is we can approximate y at t_{n+1} via the formula, $y_{n+1} \approx y_n + \Delta t f(t_n, y_n)$. The standard way to write this however is with, $h = \Delta t$, basically a renaming and we usually use $y_0 = y(t_0)$, i.e. the initial condition, and we also drop the $\approx$ and us $=$. So our general formula is,

$$y_{n+1} = y_n + h f(t_n, y_n); \; y_0 = y(t_0). \tag{1}$$

When debugging your codes use the following example, and make sure your values are close to mine. Your values might be ever so slightly off, but not more than say .0001.

(1) $f(t,y) = 3 + t - y$, which gives us the equation $y_{n+1} = y_n + h \cdot (3 + t_n - y_n)$ where $y_0 = 1$.

(a) Here we have $h = 0.1$, so we have the following t's. We get them just by starting at t_0 and incrementing. $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$, $t_4 = 0.4$. Then we have, $y_1 = y_0 + h \cdot (3 + t_0 - y_0) = 1 + (0.1)(3 + 0 - 1) = 1.2$, $y_2 = y_1 + h \cdot (3 + t_1 - y_1) = 1.39$, $y_3 = y_2 + h \cdot (3 + t_2 - y_2) = 1.571$, and $y_4 = y_3 + h \cdot (3 + t_3 - y_3) = 1.7439$. Lets put this in a table to make it look pretty,

n	0	1	2	3	4
t_n	0	0.1	0.2	0.3	0.4
y_n	1	1.2	1.39	1.571	1.7439

(b) Hopefully part a gave you a good idea of how we do these problems, so I'll just give the table of values I received when running my code on matlab (remember $h = 0.05$):

n	0	1	2	3	4	5	6	7	8
t_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
y_n	1	1.1	1.1975	1.2926	1.3855	1.4762	1.5649	1.6517	1.7366

(c) Here $h = 0.025$,

n	0	1	2	3	4	5	6	7	8
t	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
y	1	1.05	1.0994	1.1481	1.1963	1.2439	1.2909	1.3374	1.3833

n	9	10	11	12	13	14	15	16
t	0.225	0.25	0.275	0.3	0.325	0.35	0.375	0.4
y	1.4288	1.4737	1.5181	1.562	1.6055	1.6484	1.6910	1.7331

(d) Next we solve the equation via integrating factors to get $y = 2+t-e^{-t}$, and calculating the points gives us the following comparison,

h	$t =$	0.1	0.2	0.3	0.4
0.1	$y(t) =$	1.2	1.39	1.571	1.7439
0.05	$y(t) =$	1.1975	1.3855	1.5649	1.7366
0.025	$y(t) =$	1.1963	1.3833	1.562	1.7331
Exact	$y(t) =$	1.19516	1.38127	1.55918	1.72968

3.2 Existence and Uniqueness and the Wronskian

Last time we discussed ODEs of the form,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0.$$

Now lets look at the general case of,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = g(x).$$

Lets put this in standard form by dividing through by $p_n(x)$ and naming the new functions "q" and "f",

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = f(x). \quad (2)$$

Consider the simple ODE,

$$y' + q(x)y = f(x);\ q(x) = \begin{cases} 1 \text{ if } x \text{ is irrational,} \\ 0 \text{ if } x \text{ is rational;} \end{cases}$$

In order to solve this we would need to use integrating factors, however notice that q is not integrable (in the usual fashion), so we can't solve this - in fact it has no unique solution. So, we need conditions on $q's$ and f to guarantee that we can find a unique solution. We outline this in the next theorem, however one should proceed with caution because this only works for linear ODEs.

Theorem 1. *Consider ODE (2) with initial conditions: $y(x_0) = a_0$, $y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$. Then, if $q_{n-1}, q_{n-2}, \ldots, q_2, q_1, q_0$ are continuous on a common interval I containing x_0, the IVP has exactly one solution on I.*

Now we proceed to defining certain important ideas that we will use in our following theorems.

Definition 1. The set of functions $\{h_1, h_2, \ldots, h_{n-1}, h_n\}$ are said to be linearly independent if $c_1h_1+c_2h_2+\cdots+c_{n-1}h_{n-1}+c_nh_n \neq 0$, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n$ is said to be a linear combination of $h_1, h_2, \ldots, h_{n-1}, h_n$.

Last time we talked about superposition. We will pose it more rigorously in the next theorem. First consider the homogeneous ODE in standard form,

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = 0. \quad (3)$$

Theorem 2. *If $y_1, y_2, \ldots, y_{n-1}, y_n$ are solutions to (3), then any linear combination of y's are also solutions.*

For example, $y = c_1y_1 + c_2y_2$, $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, etc. are also solutions. Now we define what the Wronskian is, which will be a major part of this section.

Definition 3. Suppose $h_1(x), h_2(x), \ldots, h_{n-1}, h_n$ are functions with $n-1$ derivatives, then the Wronskian is defined to be the following determinant,

$$W = \begin{vmatrix} h_1 & h_2 & \cdots & h_n \\ h_1' & h_2' & \cdots & h_n' \\ h_1'' & h_2'' & \cdots & h_n'' \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(n-1)} & h_2^{(n-1)} & \cdots & h_n^{(n-1)} \end{vmatrix} \quad (4)$$

Theorem 3. *Suppose $y_1, y_2, \ldots, y_{n-1}, y_n$ are solutions to (3) on I, with the usual initial conditions, then they are linearly independent if and only if $W \neq 0$ for all $x \in I$.*

The next definition and theorem will allow us to find guaranteed linearly independent solutions, but note that these are not necessarily the only linearly independent solutions.

Definition 4. The set of all linearly independent solutions of an ODE is called the fundamental set of that ODE.

For the remaining theorems consider the second order ODE,

$$y'' + q_1(x)y' + q_0(x)y = 0. \quad (5)$$

Theorem 4. *Consider ODE (5), and let y_1, y_2 solve (5) for $x \in I$ such that $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x_0) = 0$, $y_2'(x_0) = 1$. Then, y_1, y_2 form a fundamental set of (5).*

The following theorem is a theorem we use in section 3.3.

Theorem 5. *If $y = u(t) + iv(t)$ solves (5) on I, then so does u and v independently, i.e. if $y = c_1u + ic_2v$ is a solution, so is $y = c_3u + c_4v$.*

The next theorem gives us a formula to compute the Wronskian without having to take a determinant, but it only works for second order ODEs.

Theorem 6 (Abel)**.** *The Wronskian of y_1, y_2 for (5) can be written as,*

$$W(y_1, y_2) = c\exp\left(-\int q_1(x)dx\right), \quad (6)$$

and is zero (if $c = 0$) or nonzero (if $c \neq 0$) for all $x \in I$.

Now lets do some example problems,

1) The derivatives are $2e^{2t}$ and $(-3/2)e^{-3t/2}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{3}{2}e^{2t-3t/2} - 2e^{2t-3t/2}.$$

3) The derivatives are $-2e^{-2t}$ and $e^{-2t} - 2te^{-2t}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}.$$

9) We put the ODE in standard form,

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}.$$

Notice, that this has discontinuities at $t = 0, 4$, and since we need to include the initial condition, the largest domain where a unique solution exists is $t \in (0, 4)$.

11) Again we convert the ODE into standard form,

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$$

This is discontinuous when $x = 0, 3$, so our largest domain where a unique solution containing the initial condition exists is $x \in (0, 3)$.

17) Here we have an inverse problem. We need to find a g that satisfies the Wronskian given, so lets take the Wronskian and see what we get,

$$W = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g = e^{2t}(g' - 2g) = 3e^{4t} \Rightarrow g' - 2g = 3e^{2t}$$

So we have to solve this first order ODE via integrating factor,

$$\mu = \exp\left(-\int^t 2d\tau\right) \Rightarrow \int d(e^{-2t}g) = \int 3dt \Rightarrow e^{-2t}g = 3t + C \Rightarrow g = 3te^{2t} + Ce^{2t}.$$

23) We go straight to the characteristic polynomial, $r^2 + 4r + 3 = (r+1)(r+3) = 0 \Rightarrow r = -1, -3$, so our general solution is $y = c_1e^{-x} + c_2e^{-3x}$. Now, by Theorem 4, we solve two different IVPs for this ODE: $y_1(1) = c_1e^{-1} + c_2e^{-3} = 1$ and $y_1'(1) = -c_1e^{-1} - 3c_2e^{-3} = 0$. By summing the two equations we get $-2c_2e^{-3} = 1 \Rightarrow c_2 = -e^3/2$, this gives $c_1 = 3e/2$, so our first solution is $y_1 = \frac{3}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}$. For the second solution we have $y_2(1) = c_1e^{-1} + c_2e^{-3} = 0$ and $y_2'(1) = -c_1e^{-1} - 3c_2e^{-3}$. We easily get $c_2 = -e^3/2$ and then $c_1 = e/2$, which gives us a solution of $y_2 = \frac{1}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}$. So, the following equations make a fundamental set of the ODE,

$$y_1 = \frac{3}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)};\ y_2 = \frac{1}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}.$$

27) For the first solution we have $y_1' = 1 \Rightarrow y_1'' = 0 \Rightarrow -xy_1' + y_1' = 0$. For the second solution we have $y_2' = \cos x \Rightarrow y_2'' = -\sin x$, then $(1 - x\cot x)(-\sin x) - x\cos x + \sin x = -\sin x + x\cos x - x\cos x + \sin x = 0$. Now, we take the Wronskian of these,

$$W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0 \text{ for } x \in (0, \pi).$$

So, they are linearly independent on that domain.

30) We put the ODE into standard form: $y'' + (\tan t)y' - ty/\cos t = 0$. Then we use Abel's theorem to get,

$$W = c\exp\left(-\int(\tan t)dt\right) = c\cos t.$$

3.3 Complex Roots

Again consider the ODE,

$$ay'' + by' + cy = 0, \tag{7}$$

which has the characteristic polynomial equation,

$$ar^2 + br + c = 0. \tag{8}$$

Using the quadratic formula we get,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What if $b^2 - 4ac < 0$? Then r is of the form $r = \xi \pm i\theta$ where $\xi, \theta \in \mathbb{R}$, but this means r is a complex conjugate. However, we do the same thing as usual to get,

$$y = c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{(\xi+i\theta)x} + c_2e^{(\xi-i\theta)x} = e^{\xi x}\left(c_1e^{i\theta x} + c_2e^{-i\theta x}\right).$$

We need to deal with the part inside the parentheses, and we do this by what's called, *Euler's Identity.* And we can derive this fairly easily by using Taylor series, since we know the taylor series,

$$e^{it} = \sum_{n=0}^{\infty}\frac{(it)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^nt^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^{n+1}t^{2n+1}}{(2n+1)!} = \cos t + i\sin t. \tag{9}$$

Then our solution becomes,

$$y = e^{\xi x}[c_1(\cos\theta x + i\sin\theta x) + c_2(\cos\theta x - i\sin\theta x)] = e^{\xi x}[(c_1+c_2)\cos\theta x + i(c_1-c_2)\sin\theta x].$$

However, we only want real solutions. Notice that $\cos\theta x$ and $\sin\theta x$, with the proper constant coefficients, are solutions to (7) independently. So, by Theorem 5, $y = e^{\xi x}(A\cos\theta x + B\sin\theta x)$ is also a solution. We have just developed a theorem,

Theorem 7. *If (8) has complex roots, i.e. $r = \xi + i\theta$, then the general solution of (7) is,*

$$y = e^{\xi x}(A\cos\theta x + B\sin\theta x). \tag{10}$$

Now, lets do some examples,

4) Applying Euler's identity, $e^{2-i\pi/2} = e^2(\cos\pi/2 - i\sin\pi/2) = -ie^2$.
6) $\frac{1}{\pi}e^{i2\ln\pi} = \frac{1}{\pi}(\cos(2\ln\pi) + i\sin(2\ln\pi))$.
10) We go to the characteristic polynomial, $r^2+2r+2 = 0$ and use the quadratic formula, $r = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$, which gives us a general solution of $y = e^{-t}(A\cos t + B\sin t)$.

18) Again our characteristic polynomial gives, $r^2+4r+5=0$, and the quadratic formula gives, $r = (-4 \pm \sqrt{-4})/2 = -2 \pm i$, so our general solution is $y = e^{-2t}(A\cos t + B\sin t)$. Now we go to our initial conditions: $y(0) = A = 1$. Then, $y'(t) = -2e^{-2t}(\cos t + B\sin t) + e^{-2t}(-\sin t + B\cos t)$, so $y'(0) = -2 + B = 0 \Rightarrow B = 2$. So, our solution is $y = e^{-2t}(\cos t + 2\sin t)$.

20) As per usual we have $r^2+1=0 \Rightarrow r = \pm i \Rightarrow y = A\cos t + B\sin t$. From the first initial condition we have, $y(\pi/3) = A/2 + \sqrt{3}B/2 = 2 \Rightarrow A = 4 - \sqrt{3}B$. From the second initial condition we have, $y'(\pi/3) = -\sqrt{3}A/2 + B/2 = -4 \Rightarrow -\sqrt{3} + 3B/2 + B/2 = -2\sqrt{3} + 2B = -4 \Rightarrow B = \sqrt{3} - 2$. So, we get $A = 1 + 2\sqrt{3}$. Then our solution is $y = (1+2\sqrt{3})\cos t + (\sqrt{3}-2)\sin t$.