(1) Sketch the direction field for $y' = y(y^2 - 4)$ and state what happens as $t \to \infty$.
Solution: $y' = 0 \Rightarrow y = 0, \pm 2$. $y' < 0$ for $y \in (-\infty, -2)$ and $y \in (0, 2)$. $y' > 0$ for $(\infty, 2)$ and $(-2, 0)$.

(2) Sketch the direction field for $y' = y(y-2)^2$.
Solution: $y' = 0 \Rightarrow y = 0, 2$. $y' < 0$ for $y \in (-\infty, 0)$. $y' > 0$ for $y \in (0, 2)$ and $y \in (2, \infty)$.

(3) Consider the ODE $2t^2 y'' + 3ty' - y = 0$.
(a) What is the order of the equation? Is it linear or nonlinear?
Solution: 2ND order linear.
(b) Is $y_1 = t^{1/2}$ a solution?
Solution: $y_1' = t^{-1/2}/2$, $y_1'' = -t^{-3/2}/4$. Plugging it in gives, $2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} = 0$. So, y_1 is a solution.
(c) Is $y_2 = t^{-1}$ a solution?
Solution: $y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$. Plugging it in gives, $2t^2(2t^{-3}) + 3t(-t^2) - t^{-1} = 0$. So, y_2 is a solution.

(4) What is the order of the ODE $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = \dfrac{\ln x}{xy}$. Is it linear or nonlinear?
Solution: 2ND order nonlinear.

(5) Solve the IVP $\dfrac{dy}{dx} = \dfrac{x}{y(1+x^2)}$, $y(0) = -2$.
Solution: We solve this via separation, so

$$\int y dy = \int \frac{x dx}{x^2+1} \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}\ln(1+x^2) + C_0 \Rightarrow y = \pm\sqrt{\ln(1+x^2) + C_1}.$$

From the initial condition we get that $C_1 = 4$ and we have to choose the negative branch since $y(0) = -2$, then our solution the the IVP is,

$$y = \pm\sqrt{\ln(1+x^2) + 4}$$

(6) Solve the IVP $y' = y^2 - 1$, $y(0) = -2$.
Solution: We solve this via separation, so

$$\int \frac{dy}{y^2-1} = \int dx \Rightarrow \frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy \Rightarrow \ln\left|\frac{y-1}{y+1}\right| = 2x + C \Rightarrow \left|\frac{y-1}{y+1}\right| = ke^{2t}.$$

Since $y(0) = -2$, $y'(0) = y(0)^2 - 1 = 3 > 0$, so the solution must be between $y = -2$ and $y = -1$ (where the left hand side blows up) and within this domain the value inside the absolute value is always positive, so we can take out the absolute values. Then we get,

$$y - 1 = ke^{2t}y + ke^{2t} \Rightarrow y = \frac{1 + ke^{2t}}{1 - ke^{2t}}$$

From the initial condition we get $k = 3$, so the solution becomes,

$$y = \frac{1 + 3e^{2t}}{1 - 3e^{2t}}$$

(7) Solve the IVP $y' + y = e^{-t}$, $y(0) = y_0$. Find the value of y_0 such that the solution $y(t)$ reaches its maximum at $t = 4$.

Solution: We solve this via integrating factor: $\mu = e^t$, then

$$\int d(e^t y) = \int dt \Rightarrow e^t y = t + C \Rightarrow y = te^{-t} + Ce^{-t}.$$

The initial condition gives us, $C = y_0$, so the solution is $y = te^{-t} + y_0 e^{-t}$. For the next part we want to find what y_0 has to be in order for y to have a maximum at $t = 4$. So, lets plug in $t = 4$ to the derivative and equate it to 0, then solve for y_0.

$$y' = e^{-t} - y = e^{-t} - te^{-t} - y_0 e^{-t} \Rightarrow y'(4) = e^{-4} - 4e^{-4} - y_0 e^{-4} = -(y_0 + 3)e^{-4} = 0 \Rightarrow y_0 = -3.$$

Just check to make sure it is in fact a maximum not a minimum using the usual calculus tools.

(8) Solve the IVP $ty' + 2y = 4t^2$, $y(1) = 4$.

Solution: First put it in standard form: $y' + (2/t)y = 4t$. We solve this via integrating factor: $\mu = \exp\left(\int (2/t)dt\right) = \exp(2\ln t) = \exp(\ln t^2) = t^2$. Then,

$$\int d(t^2 y) = \int 4t^3 dt \Rightarrow t^2 y = t^4 + C \Rightarrow y = t^2 + Ct^{-2}.$$

The initial condition gives us $C = 3$, then the solution to the IVP is $y = t^2 + 3t^{-2}$.

(9) Use Euler's method to approximate the solution of $y' = -y + 1 - t$, $y(0) = 1$ at $t = 0.2$, for $h = 0.1$.

Solution: Our formula for this problem is going to $y_{n+1} = y_n + h \cdot (-y_n + 1 - t_n)$. Then we can make the following table:

n	t_n	y_n
0	0	1
1	0.1	1
2	0.2	0.99

(10) Find a linear homogeneous constant coefficient ODE that has the roots of its characteristic equation as $r = -2, 3$.

Solution: The characteristic equation is: $(r + 2)(r - 3) = r^2 - r - 6 = 0$. Then the ODE is $y'' - y' - 6y = 0$.

(11) Solve the IVP $y'' - y' - 2y = 0$; $y(0) = \alpha$, $y'(0) = \beta$. What relation between α and β will give us a bounded (for all time) solution?

Solution: The characteristic equation is $r^2 - r - 2 = (r - 2)(r + 1) = 0 \Rightarrow r = 2, -1$. Then our general solution is $y = c_1 e^{-t} + c_2 e^{2t}$. The first initial condition gives us $y(0) = c_1 + c_2 = \alpha \Rightarrow c_2 = \alpha - c_1$. Then $y = c_1 e^{-t} + (\alpha - c_1)e^{2t} \Rightarrow y' = -c_1 e^{-t} + 2(\alpha - c_1)e^{2t}$. The second initial condition gives us $y'(0) = -c_1 + 2(\alpha - c_1) = \beta \Rightarrow c_1 = (2\alpha - \beta)/3$. Notice in order for $y(t)$ to stay bounded for all time, c_2 needs to be zero, and $c_2 = (\alpha + \beta)/3$, so we need $\beta = -\alpha$.

(12) A tank with a capacity of 6 L initially contains 10 g of salt and 1 L of water. A mixture containing 1 g/L of salt enters the tank at a rate of 3 L/hr. The mixture leaves the tank at a rate of $\frac{1}{2}V(t)$ L/hr, where $V(t)$ is the volume of fluid in the tank (which may be less than the volume of the tank itself).

(a) Formulate the IVP for the volume of fluid in the tank then solve that IVP.

Solution: This problem is similar to related rate problems that you've seen in Calc I, except in the context of ODEs. The problem is basically solved for you. You know the volume rate in is 3 L/hr and you know the volume rate out is $\frac{1}{2}V(t)$ L/hr. You also know that there is initially 1 L of fluid. Then,

$$dV/dt = 3 - V/2;\ V(0) = 1.$$

We can solve this IVP via separation,

$$\int \frac{dV}{3 - V/2} = \int dt \Rightarrow -2\ln|3-V/2| = t+C_0 \Rightarrow \ln|3-V/2| = -t/2+C_1 \Rightarrow 3-V/2 = k_0e^{-t/2} \Rightarrow V = ke^{-t/2}+6.$$

From the initial condition we get $V(0) = k + 6 = 1 \Rightarrow k = -5$, so the solution to our IVP is $V = 6 - 5e^{-t/2}$.

(b) Formulate an ODE for the amount of salt in the tank. Show that this ODE is the same as the ODE for the volume above.

Solution: This is more like our usual tank problem. The rate in for the amount of salt, x, is $(1g/L) \times (3L/hr) = 3g/hr$. The rate out is $(x/V g/L) \times (V/2L/hr) = x/2g/hr$, then our ODE is,

$$dx/dt = 3 - x/2,$$

which is the same as that for the volume for this problem.

(13) A tank initially contains 120 L of fresh water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at a rate of 3 L/min.

(a) When will the tank be empty (i.e. come up with an IVP for the volume).

Solution: This is similar to the previous problem. For the volume, rate in is 2 L/min, and the rate out is 3 L/min, so the IVP is

$$dV/dt = -1;\ V(0) = 120.$$

We can solve this via separation to get $V = -t + 120$, so the tank will be empty in 120 minutes.

(b) Formulate an IVP for the amount of salt in the tank.

Solution: For the amount salt, x, the rate in is $(\gamma g/L) \times (2L/min) = 2\gamma g/min$, and the rate out is $(x/V g/L) \times (3L/min) = 3x/V g/min = 3x/(120 - t)$. Then the IVP is

$$\frac{dx}{dt} = 2\gamma - \frac{3x}{(120 - t)};\ x(0) = 0.$$

This cannot be separated, but if we put it in standard form we will see that we can use integrating factors

$$\frac{dx}{dt} + \frac{3}{(120 - t)}x = 2\gamma \Rightarrow \mu = \exp\left(\int^t \frac{3}{120 - s}ds\right) = \exp\left(-3\ln|120 - t|\right) = (120 - t)^{-3}.$$

Then

$$\int d\left((120 - t)^{-3}x\right) = \int (120 - t)^{-3} \cdot 2\gamma dt \Rightarrow (120 - t)^{-3}x = 2\gamma \int (120 - t)^{-3}dt = \gamma(120 - t)^{-2} + C$$
$$\Rightarrow x = \gamma(120 - t) + C(120 - t)^3.$$

From the initial condition we have $x(0) = 120\gamma + 120^3C = 0 \Rightarrow C = \frac{-\gamma}{120^2}$. Then the solution to the IVP is

$$x(t) = \gamma(120 - t) - \gamma\frac{(120 - t)^3}{120^2}.$$