(1) (a) $y' = 0 \Rightarrow y_* = 0, -1, 4$.
(b) I didn't feel like drawing the direction field on MatLab, but hopefully you get the idea.

$y > 4$	$y' < 0$
$0 < y < 4$	$y' > 0$
$-1 < y < 0$	$y' < 0$
$y < -1$	$y' > 0$

(2) (a) $xy'' + y' = \ln(xy)$ 2nd order, nonlinear.
(b) $y_1' = -2\sin 2t$, $y_1'' = -4\cos 2t$, $-4\cos 2t + 4\cos 2t = 0\checkmark$.
$y_2' = -2\cos 2t$, $y_2'' = 4\sin 2t$, $4\sin 2t - 4\sin 2t = 0\checkmark$.

(3) (a) Use separation,

$$\int \frac{dy}{1-y^2} = 2\int (1+x)dx \Rightarrow \frac{1}{2}\int \left[\frac{1}{1-y} + \frac{1}{1+y}\right] dy = 2x + x^2 + C_0$$
$$\Rightarrow \frac{1}{2}[-\ln|1-y| + \ln|1+y|] = 2x + x^2 + C_0 \Rightarrow \ln\left|\frac{1+y}{1-y}\right| = 4x + 2x^2 + C_1.$$

Notice that $y(0) = -2$, so $(1+y(0))/(1-y(0)) = -1/3 < 0$. However, due to the absolute values we get a positive expression; i.e.

$$\ln\left|\frac{1+y}{1-y}\right| = \ln\left(\frac{y+1}{y-1}\right) \Rightarrow \frac{y+1}{y-1} = ke^{4x+2x^2} \Rightarrow k = \frac{1}{3}.$$

Then,

$$y + 1 = yke^{4x+2x^2} - ke^{4x+2x^2} \Rightarrow \left[1 - ke^{4x+2x^2}\right] y = -1 - ke^{4x+2x^2}$$
$$y = \frac{-1 - ke^{4x+2x^2}}{1 - ke^{4x+2x^2}} = \frac{-1 - (1/3)e^{4x+2x^2}}{1 - (1/3)e^{4x+2x^2}} = \frac{-3 - e^{4x+2x^2}}{3 - e^{4x+2x^2}}$$

As $x \to \infty$ we run into a snag. The solution asymptotes at $\exp(4x + 2x^2) = 3$ and there are no solutions past this point. The asymptote is

$$4x+2x^2 = \ln 3 \Rightarrow x^2+2x = \frac{1}{2}\ln 3 \Rightarrow x^2+2x+1 = \frac{1}{2}\ln 3+1 \Rightarrow x+1 = \pm\sqrt{\frac{1}{2}\ln 3 + 1} \Rightarrow x = -1\pm\sqrt{\frac{1}{2}\ln 3 + 1}.$$

Choose the positive branch since $x > 0$.
(b) $\mu = e^{-t}$, so

$$\int d\left(e^{-t}y\right) = -\frac{1}{2}\int dt \Rightarrow e^{-t}y = -\frac{1}{2}t + C \Rightarrow y = -\frac{1}{2}te^t + Ce^t.$$

$y(0) = C = y_0 \Rightarrow y = -te^t/2 + y_0e^t$.

$$y' = -\frac{1}{2}te^t + \left(y_0 - \frac{1}{2}\right)e^t \Rightarrow y'(2) = -e^2 + \left(y_0 - \frac{1}{2}\right)e^2 = 0 \Rightarrow y_0 - \frac{1}{2} = 1 \Rightarrow y_0 = \frac{3}{2}.$$

(4) The equation volume is,

$$\frac{dV}{dt} = 2;\ V(0) = 1 \Rightarrow V = 2t + C,\ V(0) = C = 1 \Rightarrow V = 1 + 2t.$$

So, $V = 9 \Rightarrow t = 4$.

The equation for salt is,

$$\frac{dx}{dt} = 4 - \frac{2x}{1+2t} \Rightarrow \frac{dx}{dt} + \frac{2}{1+2t}x = 4;\ x(0) = 10.$$

Use integrating factors,

$$\mu = \exp\left(\int^t \frac{2ds}{1+2s}\right) = \exp\left(\ln|1+2t|\right) = 1 + 2t.$$

Then,

$$\int d((1+2t)x) = 4\int (1+2t)dt \Rightarrow (1+2t)x = 4t + 4t^2 + C;\ x(0) = 10 \Rightarrow C = 10 \Rightarrow x = \frac{4t^2+4t+10}{1+2t}.$$

So, $x(4) = 10 \Rightarrow \frac{x(4)}{V(4)} = \frac{10}{9}$ g/L.

(5) $y_{n+1} = y_n + h\left(y_n^2 - t_n^2\right)$.

n	t_n	y_n
1	.1	$1 + .1(1-0) = 1.1$
2	.2	$1.1 + .1(1.1^2 - .1^2) = 1.1 + .1(1.21 - .01) = 1.1 + .12 = 1.22$
3	.3	$1.22 + .1(1.22^2 - .2^2)$.

(6) (a) $(r+1)(r-3) = r^2 - 2r - 3 \Rightarrow y'' - 2y' - 3 = 0$. The solution is $y = c_1e^{-t} + c_2e^{3t}$. Plugging in the first initial condition gives us,

$$y(0) = c_1 + c_2 = \alpha \Rightarrow c_2 = \alpha - c_1 \Rightarrow y = c_1e^{-t} + (\alpha - c_1)e^{3t}.$$

The second initial condition gives us,

$$y' = -c_1e^{-t} + 3(\alpha - c_1)e^{3t} \Rightarrow y'(0) = -c_1 + 3(\alpha - c_1) = \beta \Rightarrow -4c_1 + 3\alpha = \beta \Rightarrow c_1 = \frac{3\alpha - \beta}{4}.$$

The full solution is,

$$y = \frac{3\alpha - \beta}{4}e^{-t} + \left(\alpha - \frac{3\alpha - \beta}{4}\right)$$

If we get rid of the second term the solution stays bounded for all time, so

$$\alpha - \frac{3\alpha - \beta}{4} = 0 \Rightarrow 4\alpha = 3\alpha - \beta \Rightarrow \alpha = -\beta.$$

(b) $r^2 - 4 = 0 \Rightarrow r = \pm 2 \Rightarrow y = c_1e^{-2t} + c_2e^{2t}$. The first initial condition gives us,

$$y(0) = c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1 \Rightarrow y = c_1e^{-2t} + (1 - c_1)e^{2t}.$$

The second initial condition gives us,

$$y' = -2c_1e^{-2t} + 2(1 - c_1)e^{2t} \Rightarrow y'(0) = -2c_1 + 2 - 2c_1 = 0 \Rightarrow c_1 = \frac{1}{2} \Rightarrow y = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t}.$$

I'm not going to draw it, but the only thing you have to find is the min: $(x, y) = (0, 1)$, and it's easy to sketch after that.