(1) Consider the IVP: $t(t-4)y'' - 3ty' + 4y = 2;\ y(3) = 0,\ y'(3) = -1$.

(a) Please determine the longest interval for which the IVP is guaranteed to have a unique solution.

Solution: The discontinuities are at $t = 0, 4$. The initial condition is $t = 3$, so $\boxed{t \in (0,4)}$.

(b) Please compute the Wronskian using Abel's theorem.

Solution:

$$W(y_1, y_2) = C\exp\left(3\int \frac{dt}{t-4}\right) = C\exp\left(3\ln|t-4|\right).$$

Notice that $t - 4 < 0$ for $t \in (0,4)$, so $|t-4| = 4 - t$, then

$$W = C\exp\left(\ln(4-t)^3\right) = \boxed{C(4-t)^3}.$$

(2) Consider the IVP: $4y'' + 12y' + 9y = 0;\ y(0) = -1,\ y'(0) = \alpha$.

Solution: $4r^2 + 12r + 9 = 0 \Rightarrow r^2 + (12/4)r + (3^2/2^2) = 0 \Rightarrow (r + 3/2)^2 = 0 \Rightarrow \boxed{r = -3/2} \Rightarrow \boxed{y = (c_1 + c_2 t)e^{-3t/2}}$. The first initial conditions give us $y(0) = \boxed{c_1 = -1} \Rightarrow y = (-1 + c_2 t)e^{-3t/2}$. For the second initial condition lets take the derivative, $y' = c_2 e^{-3t/2} - \frac{3}{2}(-1 + c_2 t)e^{-3t/2}$, so $y'(0) = c_2 + 3/2 = \alpha \Rightarrow \boxed{c_2 = \alpha - 3/2}$, then the solution is $\boxed{y = [-1 + (\alpha - 3/2)t]e^{-3t/2}}$.

(a) For what α does the solution change signs at $t = 1/2$?

Solution: A change of sign always happens around the time the curve hits zero,

$$t = \frac{1}{2} \Rightarrow y = \left(-\frac{7}{4} + \frac{\alpha}{2}\right)e^{-3/4} = 0 \Rightarrow \frac{\alpha}{2} = \frac{7}{4} \Rightarrow \alpha = \boxed{\frac{7}{2}}.$$

(b) How many times does this solution (for the α above) change signs for $t > 0$?

Solution: Same idea as above,

$$\alpha = \frac{7}{2} \Rightarrow y = (-1 + 2t)e^{-3t/2} = 0 \Rightarrow t = \frac{1}{2}.$$

Since we only have one solution for t, y will change signs only once.

(3) One solution to $t^2y'' - 3ty' + 4y = 0$ is $y_1 = t^2$. Please find the other solution.

Solution: Let $y = v(t)y_1 \Rightarrow y' = v'y_1 + vy_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$. Plugging this into the ODE gives

$$t^2vy_1'' + 2t^2v'y_1' + t^2vy_1'' - 3tv'y_1 - 3tvy_1' + 4vy_1 = t^2v''y_1 + (2t^2y_1' - 3ty_1)v' + \cancelto{0}{(t^2y_1'' - 3ty_1' + 4y_1)}v = 0.$$

Now let $u = v'$, then

$$t^2y_1u' = (3ty_1 - 2t^2y_1')u \Rightarrow t^4u' = (3t^3 - 4t^3)u \Rightarrow u' = -\frac{1}{t}u$$

$$\Rightarrow \int\frac{du}{u} = -\int\frac{dt}{t} \Rightarrow \ln u = -\ln t + c_1 \Rightarrow \boxed{u = \frac{k}{t}} \Rightarrow \boxed{v = k\ln t + c_2}$$

$$\Rightarrow y = v(t)y_1 = \boxed{kt^2\ln t + c_2t^2} \Rightarrow \boxed{y_2 = t^2\ln t}.$$

(4) One solution to $2t^2y'' + 3ty' - y = 0$ is $y_1 = 1/t$. Please find the other solution.

Solution: Let $y = v(t)y_1 \Rightarrow y' = v'y_1 + vy_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$. Plugging this into the ODE gives

$$2t^2v''y_1 + 4t^2v'y_1' + 2t^2vy_1'' + 3tv'y_1 + 3tvy_1' - vy_1 = 2t^2v''y_1 + (4t^2y_1' + 3ty_1)v' + \cancelto{0}{(2t^2y_1'' + 3ty_1' - y_1)}v = 0.$$

Let $u = v'$,

$$2t^2u'y_1 = -(4t^2y_1' + 3ty_1)u \Rightarrow 2tu' = u \Rightarrow \int \frac{du}{u} = \frac{1}{2}\int \frac{dt}{t}$$
$$\Rightarrow \ln u = \frac{1}{2}\ln t + c_1 \Rightarrow \boxed{u = k_1\sqrt{t}} \Rightarrow \boxed{v = k_2t^{3/2} + c_2}$$
$$\Rightarrow \boxed{y = k_2t^{1/2} + c_2\frac{1}{t}} \Rightarrow \boxed{y_2 = \sqrt{t}}.$$

(5) Please use the method of undetermined coefficients to find the form of the particular solution (**WITHOUT SOLVING FOR CONSTANTS**) of the following ODEs.

(a)

$$y'' + 5y' + 6y = -t + e^{-3t} + te^{-2t} + e^{-3t}\cos t$$

Solution: The characteristic solution is $y_c = c_1e^{-2t} + c_2e^{-3t}$. We take a guess at the particular solution using our forcing function

$$y_p \stackrel{?}{=} a_0 + a_1t + ke^{-3t} + (b_0 + b_1t)e^{-2t} + e^{-3t}[A_1\cos t + A_2\sin t].$$

Notice that there is a repeat with e^{-3t} and e^{-2t}, so our particular solution becomes

$$y_p = a_0 + a_1t + kte^{-3t} + t(b_0 + b_1t)e^{-2t} + e^{-3t}[A_1\cos t + A_2\sin t].$$

(b)

$$y'' + 3y' + 2y = e^t(t^2 + 1)\sin(2t) + 3e^{-t}\cos t + 4e^t.$$

Solution: The characteristic solution is $y_c = c_1e^{-2t} + c_2e^{-t}$. We take a guess at our particular solution, but as we'll see, we won't have any repeats,

$$y_p = (a_0 + a_1t + a_2t^2)e^t[A\cos 2t + A_2\sin 2t] + e^{-t}[B_1\cos t + B_2\sin t] + b_0e^t.$$

(6) Please find the general solution of the ODE: $y'' + 4y' + 4y = t^{-2}e^{-2t};\ t > 0$

Solution: The characteristic solution is $y_c = (c_1 + c_2t)e^{-2t}$, so our two solutions are $\boxed{y_1 = e^{-2t}}$ and $\boxed{y_2 = te^{-2t}}$. The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = \boxed{e^{-4t}}.$$

Then we use our formula for variation of parameters

$$y = -y_1\int \frac{y_2f(t)}{W(y_1,y_2)}dt + y_2\int \frac{y_1f(t)}{W(y_1,y_2)}dt = -e^{-2t}\int \frac{te^{-2t}\cdot t^{-2}e^{-2t}}{e^{-4t}}dt + te^{-2t}\int \frac{e^{-2t}\cdot t^{-2}e^{-2t}}{e^{-4t}}dt$$
$$= \boxed{-e^{-2t}[\ln t + c_3] + te^{-2t}\left[-t^{-1} + c_4\right]}.$$

Since they ask for the general solution, there's no need to simplify it any further.

(7) Consider the ODE $y'' + 2y' + 2y = \cos t$.

(a) Please find the general solution.

Solution: The characteristic polynomial is $r = \frac{1}{2}(-2 \pm \sqrt{4-8}) = \boxed{-1 \pm i}$, then $y_c = e^{-t}[A_1 \cos t + A_2 \sin t]$. We'll see that there won't be any repeats, so the particular solution is

$$y_p = B_1 \cos t + B_2 \sin t. \Rightarrow y_p' = -B_1 \sin t + B_2 \cos t \Rightarrow y_p'' = -B_1 \cos t - B_2 \sin t.$$

Plugging it into the ODE gives

$$- B_1 \cos t - B_2 \sin t - 2B_1 \sin t + 2B_2 \cos t + 2B_1 \cos t + 2B_2 \sin t = \cos t$$

$$\Rightarrow (B_1 + 2B_2) \cos t + (B_2 - 2B_1) \sin t = \cos t \Rightarrow B_2 = 2B_1 \Rightarrow \boxed{B_1 = \frac{1}{5}}, \boxed{B_2 = \frac{2}{5}}$$

$$\Rightarrow \boxed{y = e^{-t}[A_1 \cos t + A_2 \sin t] + \frac{1}{5} \cos t + \frac{2}{5} \sin t}.$$

(b) What happens to the solution as $t \to \infty$?

Solution: As $t \to \infty$, $y \to \frac{1}{5} \cos t + \frac{2}{5} \sin t$

(8) Consider the ODE $2t^2 y'' - ty' + y = t\sqrt{t}$.

(a) One solution to the homogeneous ODE is $y_1 = t$. Use reduction of order to find the other solution y_2.

Solution: Let $y = v(t)y_1 \Rightarrow y' = v'y_1 + vy_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$. Plugging this into the ODE gives

$$2t^2 v'' y_1 + 4t^2 v' y_1' + 2t^2 v y_1'' - tv' y_1 - tv y_1' + v y_1 = 2t^2 v'' y_1 + (4t^2 y_1' - t y_1) v' + \cancelto{0}{(2t^2 y_1'' - t y_1' + y_1)} v = 0.$$

Let $u = v'$

$$2t^2 y_1 u' = (t y_1 - 4t^2 y_1')u \Rightarrow 2t^{\cancel{3}} u' = -3\cancel{t^2} u \Rightarrow \int \frac{du}{u} = -\frac{3}{2} \int \frac{dt}{t}$$

$$\Rightarrow \ln u = -\frac{3}{2} \ln t + c_1 \Rightarrow \boxed{u = k_1 t^{-3/2}} \Rightarrow \boxed{v = k_2 t^{-1/2} + c_2}$$

$$\Rightarrow \boxed{y = k_2 t^{1/2} + c_2 t} \Rightarrow \boxed{y_2 = t^{1/2}}.$$

(b) Use the characteristic solution $y_c = c_1 y_2 + c_2 y_2$ to find the general solution to the full ODE.

Solution: The forcing function is $\boxed{f(t) = \frac{1}{2} t^{-1/2}}$ (i.e. standard form) and Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} t & t^{1/2} \\ 1 & \frac{1}{2} t^{-1/2} \end{vmatrix} = \boxed{-\frac{1}{2} t^{1/2}}$$

Then we plug into our formula to get

$$y = -y_1 \int \frac{y_2 f(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f(t)}{W(y_1, y_2)} dt = -t \int \frac{t^{1/2} \cdot (1/2) t^{-1/2}}{-\frac{1}{2} t^{1/2}} dt + t^{1/2} \int \frac{t \cdot (1/2) t^{-1/2}}{-\frac{1}{2} t^{1/2}} dt$$

$$= \boxed{-t \left[-2t^{1/2} + c_3\right] + t^{1/2}[-t + c_4]}$$

Again, since only the general solution was required there is no need to simplify any further.

(9) Please solve the IVP: $y'' + 4y = 6\sin(4t)$; $y(0) = y'(0) = 0$.

Solution: The characteristic solution is $y_c = A_1 \cos 2t + A_2 \sin 2t$, and we'll see that there are no repeats so the particular solution is

$$y_p = B_1 \cos 4t + B_2 \sin 4t \Rightarrow y_p' = -4B_1 \sin 4t + 4B_2 \cos 4t \Rightarrow y_p'' = -16B_1 \cos 4t - 16B_2 \sin 4t$$

Plugging it into the ODE gives

$$-16B_1 \cos 4t - 16B_2 \sin 4t + 4B_1 \cos 4t + 4B_2 \sin 4t = -12B_1 \cos 4t - 12B_2 \sin 4t = 6\sin 4t$$

$$\Rightarrow \boxed{B_1 = 0, B_2 = -\frac{1}{2}} \Rightarrow y = A_1 \cos 2t + A_2 \sin 2t - \frac{1}{2}\sin 4t.$$

The first initial condition gives $\boxed{y(0) = A_1 = 0}$, and the second initial condition gives $y'(0) = 2A_2 - 2 = 0 \Rightarrow \boxed{A_2 = 1}$, then our solution is

$$\boxed{y = \sin 2t - \frac{1}{2}\sin 4t}$$

(10) Consider the IVP $y'' - 3y' - 4y = t + 2$; $y(0) = 3$, $y'(0) = 0$.

(a) Please find the solution to the IVP.

Solution: The characteristic solution is $y_c = c_1 e^{-4t} + c_2 e^{-t}$, and we'll see that there are no repeats so our particular solution is $y_p = a_1 t + a_0 \Rightarrow y_p' = a_1 \Rightarrow y_p'' = 0$. Plugging into the ODE gives

$$-3a_1 - 4a_1 t - 4a_0 = t + 2 \Rightarrow \boxed{a_1 = -\frac{1}{4}} \Rightarrow \boxed{a_0 = -\frac{5}{16}} \Rightarrow y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{4}t - \frac{5}{16}.$$

The first initial condition gives us $y(0) = c_1 + c_2 - 5/16 = 3 \Rightarrow c_1 + c_2 = 53/16$. The derivative of the solution is $y' = 4c_1 e^{4t} - c_2 e^{-t} - 1/4$, then the second initial condition gives us $y'(0) = 4c_1 - c_2 - 1/4 = 0 \Rightarrow c_1 - c_2/4 = 1/16$. Then we get $\boxed{c_2 = 13/5}$ and $\boxed{c_1 = 57/80}$ and our solution becomes

$$\boxed{y = \frac{57}{80}e^{4t} + \frac{13}{5}e^{-t} - \frac{1}{4}t - \frac{5}{16}}.$$

(b) What happens to the solution as $t \to \infty$?

Solution: As $t \to \infty$, $y \to \infty$.

(11) A mass weighing 1/2 lb (i.e. mass $= 1/64 lb \cdot s^2/ft$) stretches a spring 1/2 ft.

(a) Suppose the system has no damping. The mass is initially pulled down 1/2 ft and released.

(i) Write down the IVP for this system.

Solution: $\boxed{k = F/x = \frac{1/2}{1/2} = 1}$, so our IVP is

$$\frac{1}{64}x'' + x = 0;\ x(0) = \frac{1}{2},\ x'(0) = 0.$$

(ii) Solve the IVP. The general solution will be $\boxed{x = A\cos 8t + B\sin 8t}$. The initial conditions give us $x(0) = \boxed{A = 1/2}$ and $x'(0) = \boxed{8B = 0}$. Then the solution is

$$\boxed{x = \frac{1}{2}\cos 8t}.$$

(iii) When does the mass return to the equilibrium position (i.e. $x = 0$).

Solution: $x = 0 \Rightarrow \boxed{t = \pi/16}$ for the first time.

(b) Now suppose the system has a damping constant of $2lb \cdot s/ft$. The mass is initially pushed up 1/2 ft and released with a downward velocity of 1/2 ft/s.

(i) Write down the IVP for this system.

Solution: The damping adds a $2x'$ term, so

$$\frac{1}{64}x'' + 2x' + x = 0;\ x(0) = -\frac{1}{2},\ x'(0) = \frac{1}{2}.$$

(ii) Solve the IVP.

Solution: This is where the problem starts to be a pain, but basically, the roots are

$$r^2 + 128r + 64 = 0 \Rightarrow r = \frac{1}{2}\left(-128 \pm \frac{1}{2}\sqrt{128^2 - 4 \cdot 64}\right) \Rightarrow \boxed{r_{1,2} = -64 \pm 8\sqrt{63}}$$

From this point lets just write down things in the general form because it doesn't make sense to carry all those ridiculous number around.

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The first initial condition gives us $x(0) = c_1 + c_2 = -1/2$ and the second initial condition gives us $x'(0) = r_1 c_1 + r_2 c_2 = 1/2$, then we get

$$c_2 = \frac{r_1 + 1}{2(r_2 - r_1)}, \qquad c_1 = \frac{r_2 + 1}{2(r_1 - r_2)}.$$

Moving on...

(12) Given that $y_1 = 1/t$ is a solution, please find another solution to the ODE

$$t^2 y'' + 3ty' + y = 0; t > 0$$

Solution: Let $y = v(t)y_1 \Rightarrow y' = v'y_1 + vy_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$, Plugging into the ODE gives us

$$t^2 v'' y_1 + 2t^2 v' y_1' + t^2 v y_1'' + 3tv' y_1 + 3tvy_1' + vy_1 = t^2 y_1 v'' + (2t^2 y_1 + 3ty_1)v' + \underbrace{(t^2 y_1'' + 3ty_1' + y_1)}_{0} v = 0.$$

Let $u = v'$,

$$t^2 y_1 u' = -(2t^2 y_1' + 3ty_1)u \Rightarrow tu' = -(-2+3)u = -u \Rightarrow \int \frac{du}{u} = -\int \frac{dt}{t}$$

$$\Rightarrow \ln u = -\ln t + c_1 \Rightarrow \boxed{u = k_1 \frac{1}{t}} \Rightarrow \boxed{v = k_2 \ln t + c_2}$$

$$\Rightarrow \boxed{y = k_2 \frac{1}{t} \ln t + c_2 \frac{1}{t}} \Rightarrow \boxed{y_2 = \frac{1}{t} \ln t}$$

(13) Please solve the following IVP

$$y'' + 4y = 3\sin 2t;\ y(0) = 2,\ y'(0) = -1.$$

Solution: The characteristic solution is $y_c = A_1 \cos 2t + A_2 \sin 2t$ and our guess for the particular solution is $y_p \stackrel{?}{=} B_1 \cos 2t + B_2 \sin 2t$, but look at that, we have a repeat, so our particular solution actually is

$$y_p = B_1 t \cos 2t + B_2 t \sin 2t \Rightarrow y_p' = B_1 \cos 2t - 2B_1 t \sin 2t + B_2 \sin 2t + 2B_2 t \cos 2t$$
$$\Rightarrow y_p'' = -4B_1 \sin 2t - 4B_1 t \cos 2t + 4B_2 \cos 2t - 4B_2 t \sin 2t.$$

Plugging this into the ODE gives

$$-4B_1 \sin 2t - 4B_1 t \cos 2t + 4B_2 \cos 2t - 4B_2 t \sin 2t + 4B_1 t \cos 2t + 4B_2 t \sin 2t$$
$$= -4B_1 \sin 2t + 4B_2 \cos 2t = 3\sin 2t \Rightarrow \boxed{B_2 = 0,\ B_1 = -\frac{3}{4}}.$$

Then the general solution is

$$y = A_1 \cos 2t + A_2 \sin 2t - \frac{3}{4} t \cos 2t \Rightarrow y(0) = \boxed{A_1 = 2}$$

The derivative of this is

$$y' = -2A_1 \sin 2t + 2A_2 \cos 2t - \frac{3}{4} \cos 2t + \frac{3}{2} t \sin 2t \Rightarrow y'(0) = 2A_2 - \frac{3}{4} = -1 \Rightarrow \boxed{A_2 = -\frac{1}{8}}.$$

Then our solution is

$$\boxed{y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4} t \cos 2t.}$$

Brief Summary of Chapter 3

Obviously incomplete so make sure you read the notes as well!

- Section 3.2: Existence and Uniqueness and Wronskian ($y'' + p(x)y' + q(x)y = 0;\ y(0) = y_0,\ y'(0) = Y_0$)
  - Existence and Uniqueness: Put ODE in standard form and list intervals for which the coefficients and forcing function are continuous, then pick out the interval that contains the initial conditions.
  - Wronskian: $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.
  - Abel's Theorem: $W(y_1, y_2) = C \exp\left(-\int^x p(\xi) d\xi\right)$

- Section 3.3: Complex Roots: $r = \xi \pm i\theta \Rightarrow y = e^{\xi x}(A \cos\theta x + B \sin\theta x)$.

- Section 3.4: Repeated Roots and Reduction of Order
  - Repeated Roots: $y = (c_1 + c_2 t)e^{rt}$.
  - Reduction of Order: If y_1 is a solution, let $y = v(x)y_1$, plug it into the ODE. The $v(x)$ term disappears and you're left with v' and v'' terms. Let $u = v'$, solve that ODE, then integrate u to get v. If you kept the constants of integration multiply v by y_1 to get your general solution.

- Section 3.5: Undetermined coefficients: Find y_c. Use $f(x)$ to guess at y_p and group like terms. If there are no repeats, that's your y_p. If there are repeats, get rid of the repeats by multiplying through by x as many times as needed.

- Section 3.6: Variation of Parameters: If we know y_1 and y_2 (sometimes given, sometimes found by solving the homogeneous equation), then $y = -y_1 \int \frac{y_2 f(x)}{W(y_1,y_2)} dt + y_2 \int \frac{y_1 f(x)}{W(y_1,y_2)} dt$.

- Section 3.7: Applications without forcing: Know how to set up and solve the IVPs. The amplitude in undamped oscillatory motion is the constant: $R = \sqrt{A^2 + B^2}$. For damped oscillatory motion $R(t) = e^{\xi t}\sqrt{A^2 + B^2}$. For undamped oscillatory motion the frequency and period are $\omega = \sqrt{k/m}$ and $T = 2\pi/\omega$. For damped oscillatory motion the quasi-frequency and quasi-period are θ (the imaginary part of the root) and $T_d = 2\pi/\theta$.

- Section 3.8: Applications with forcing: Know your trig identities. The transient solution is the one that goes to zero. The steady state solution is the one that persists. You can get resonance in an undamped system if the frequency of your forcing function is the same as the frequency of your characteristic solution because this causes a repeat and the particular solution has to be multiplied by t. You can get resonance in an undamped system if damping is low enough and the frequency of your forcing function is close enough to the "natural frequency", ω_0 (the frequency of the system in the absence of forcing and damping).

  Important identities: $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ and $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. Typically, $a = \frac{1}{2}(\omega_0 + \omega)t$ and $b = \frac{1}{2}(\omega_0 - \omega)t$ for this section.

- Trig/Hyp identities that may be useful overall:

$$\sin^2\theta + \cos^2\theta = 1;\ \cosh^2 x - \sinh^2 x = 1$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b;\ \sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos 2\theta);\ \cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos 2\theta)$$

  You can basically derive any other (seldom used) trig identity that you may need from these by either dividing through by a sin or a cos.