(1) (a) Take the Laplace and isolate the Y,

$$(s^2+s)Y = \frac{s+1}{(s+1)^2+1} \Rightarrow Y = \frac{1}{s(s^2+2s+2)}.$$

Partial fractions gives us,

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2} \Rightarrow As^2+2As+2A+Bs^2+Cs = 1 \Rightarrow (A+B)s^2+(2A+C)s+2A = 1.$$

This gives us $\boxed{A=1/2}$, $\boxed{C=-1}$, and $\boxed{B=-1/2}$,

$$Y = \frac{1/2}{s} - \frac{1}{2}\cdot\frac{s+2}{(s+1)^2+1} = \frac{1/2}{s} - \frac{1}{2}\cdot\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\cdot\frac{1}{(s+1)^2+1} \Rightarrow \boxed{y = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t}$$

(b) (i) We break up F and take the inverse Laplace,

$$F(s) = e^{-s}\frac{1}{(s+2)^2+1} - 3\frac{s+2}{(s+2)^2+1} + 6\frac{1}{(s+2)^2+1} \Rightarrow \boxed{f(t) = e^{-2(t-1)}\sin(t-1)u_1(t) - 3e^{-2t}\cos t + 6e^{-2t}\sin t}$$

(ii) There are two ways to solve this,

Method 1: If we choose to use convolutions we write G as

$$G(s) = e^{-s}\frac{3}{s^2+9}\cdot\frac{s}{s^2+9}.$$

Then the convolution is,

$$g(t) = \int_0^t u_1(\tau)\sin 3(\tau-1)\cos 3(t-\tau)d\tau = \boxed{\begin{cases} 0 & 0 \le t \le 1, \\ f(t-1) & t \ge 1; \end{cases} = f(t-1)u_1(t)}$$

where

$$f(t-1) = \int_1^t \sin 3(\tau-1)\cos 3(t-\tau)d\tau = \frac{1}{2}\int_1^t \left[\sin 3(t-1) + \sin 3(2\tau-t-1)\right]d\tau,$$

via the trig identity: $\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$. Evaluating the integral gives us,

$$\begin{aligned} f(t-1) &= \frac{1}{2}\tau\sin 3(t-1) - \frac{1}{12}\cos 3(2\tau-t-1)\Big|_1^t \\ &= \frac{1}{2}t\sin 3(t-1) - \cancel{\frac{1}{12}\cos 3(t-1)} - \frac{1}{2}\sin 3(t-1) + \cancel{\frac{1}{12}\cos 3(1-t)} \\ &= \boxed{\frac{1}{2}(t-1)\sin 3(t-1)} \end{aligned}$$

Method 2: If we choose not to use convolutions you have to recognize that G can be written as

$$G(s) = -\frac{3}{2}e^{-s}\frac{-2s}{(s^2+9)^2}.$$

Then you have to recognize that this can be written as a derivative,

$$G(s) = -\frac{3}{2}e^{-s}\frac{d}{ds}\left(\frac{1}{s^2+9}\right).$$

Then we take the inverse Laplace of what's in the parentheses,

$$\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = \frac{1}{3}\sin 3t$$

Then using # 19, we get

$$\mathcal{L}\{\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\} = -\frac{t}{3}\sin 3t$$

Then we get that the inverse Laplace Transform of g is,

$$\boxed{g(t) = \frac{1}{2}(t-1)\sin 3(t-1)u_1(t)}$$

(2) $g(t) = u_5(t) - u_{20}(t) \Rightarrow (s^2+2s+2)Y = e^{-5s}/s - e^{-20s}/s$, then isolating Y gives us

$$Y = \frac{e^{-5s}-e^{-20s}}{s(s^2+2s+2)} = (e^{-5s}-e^{-20s})\left(\frac{1/2}{s} - \frac{1}{2}\cdot\frac{s+2}{(s+1)^2+1}\right).$$

Notice that we already did the partial fractions in 1a. Then our solution is

$$\boxed{y = u_5(t)\left[\frac{1}{2} - \frac{1}{2}e^{-(t-5)}\cos(t-5) - \frac{1}{2}e^{-(t-5)}\sin(t-5)\right] - u_{20}(t)\left[\frac{1}{2} - \frac{1}{2}e^{-(t-20)}\cos(t-20) - \frac{1}{2}e^{-(t-20)}\sin(t-20)\right]}$$

(3) (a) We take the Laplace and isolate Y

$$-y'(0) - sy(0) + s^2Y + Y = G(s) \Rightarrow 1 - s + (s^2+1)Y = G(s) \Rightarrow Y = \frac{G(s)}{s^2+1} + \frac{s-1}{s^2+1}.$$

Then our solution is

$$\boxed{y = \cos t - \sin t + \int_0^t \sin\tau g(t-\tau)d\tau}$$

(b) For this part the wording was weird. There is no way easy way to evaluate ("carry out") the integral through first principles. What it should have said is "find the answer in closed form". To do that, since $g(t) = \delta(t-\pi/4) - \delta(t-\pi/2)$, $G(s) = e^{-(\pi/4)s} - e^{-(\pi/2)s}$, then our solution is

$$\boxed{y = \cos t - \sin t + u_{\pi/4}(t)\sin(t-\pi/4) - u_{\pi/2}(t)\cos(t-\pi/2)}$$

We will talk more about this in class.

(4) (a) We first compute the eigenvalues

$$\begin{vmatrix} 3-\lambda & 2 \\ -3 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda) + 6 = \lambda^2 + \lambda - 12 + 6 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3) = 0 \Rightarrow \boxed{\lambda = 2, -3}$$

Then we find the eigenvectors

$$\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} x^{(1)} = 0 \Rightarrow \boxed{x^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}; \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix} x^{(2)} = 0 \Rightarrow \boxed{x^{(2)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

Then our general solution is

$$\boxed{x = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ -3 \end{pmatrix}e^{-3t}}$$

(b) Now, if $x \to 0$ as $t \to \infty$, we need $c_1 = 0$, then

$$x(0) = \begin{pmatrix} c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 6 \end{pmatrix} \Rightarrow c_2 = \alpha \Rightarrow -3\alpha = 6 \Rightarrow \boxed{\alpha = -2}$$

(5) (a) Plugging in the ansatz gives us

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - a_nx^{n+1} = 0$$

The coefficients will be

$$x^0: \qquad 2a_2 = 0$$

$$x^{m+3}: \qquad (m+3)(m+2)a_{m+3} - a_m = 0 \Rightarrow \boxed{a_{m+3} = \frac{a_m}{(m+3)(m+2)}}$$

(b) Then we find our two solutions,

$$a_0 = 0,\ a_1 = 1:\ a_3 = 0, a_4 = \frac{1}{12} \Rightarrow \boxed{y_1 = x + \frac{1}{12}x^4} + \cdots$$

$$a_0 = 1,\ a_1 = 0:\ a_3 = \frac{1}{6} \Rightarrow \boxed{y_2 = 1 + \frac{1}{6}x^3} + \cdots$$

(6) (a) First we put this into standard form

$$y'' - \frac{x}{(1-x)(1+x)}y' + y = 0$$

Then we take our respective limits

$$\lim_{x\to 1}(x-1)\frac{x}{(x-1)(x+1)} = \frac{1}{2}\checkmark \qquad \lim_{x\to 1}(x-1)^2 = 0\checkmark$$

$$\lim_{x\to -1}(x+1)\frac{-x}{(x-1)(x+1)} = \frac{1}{2}\checkmark \qquad \lim_{x\to -1}(x+1)^2 = 0\checkmark$$

(b) The characteristic polynomial gives us

$$r(r-1) + \beta = r^2 - r + \beta = 0 \Rightarrow \boxed{r = \frac{1}{2}\left[1 \pm \sqrt{1-4\beta}\right]}.$$

There are two possibilities, either $1-4\beta < 0$, in which case it's complex conjugates with a positive real part, so $y \to 0$ as $x \to 0$. Or, $1-4\beta \geq 0$, then we require $1 \pm \sqrt{1-4\beta} > 0$, but since the positive branch will always be positive we need the negative branch to give us something positive. So, $1 - \sqrt{1-4\beta} > 0$, which means $\beta > 0$. Since for both cases we get conditions where $\boxed{\beta > 0}$, that is our final condition on β.